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Mutations in Representation Theory, Combinatorics and Homotopical Algebra.

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Contents

Chapter 0

Introduction

Ce document est un résumé des recherches que j'ai menées depuis la fin de mon doctorat. Ces travaux ont porté sur diverses thématiques : catégories triangulées, théorie des représentations de carquois, combinatoire algébrique, algèbre homotopique. Ils ont pourtant tous un fil conducteur : la notion de mutation, centrale à la définition d'algèbre amassée.

Un paradigme commun est également sous-jacent à la majorité de mes recherches ; celui de la catégorification. Il est mathématiquement pertinent d'enrichir une situation combinatoire en introduisant des espaces vectoriels dont les dimensions seront les nombres entiers apparaissant combinatoirement. Cela permet l'utilisation d'applications linéaires, offrant ainsi à la fois plus de souplesse pour étudier la situation donnée, et plus d'outils pour répondre aux questions soulevées. Plus généralement, le rôle de la catégorie des espaces vectoriels peut être joué par diverses catégories plus adaptées au problème. Dans ce mémoire, il s'agira de la catégorie des représentations d'une algèbre, le plus souvent de dimension finie sur un corps, définie par un carquois à relations ; ou des variantes de cette catégorie : catégorie dérivée, catégorie amassée, catégorie des complexes de projectifs à deux termes. Le point de vue des catégories, l'utilisation ou la construction de structures sur ces catégories sont donc très présents tout au long du mémoire.

Survol des chapitres

Les algèbres amassées sont définies par générateurs et relations mais, de façon assez inhabituelle, ces générateurs et relations ne sont pas donnés a priori. Ils sont construits récursivement, à partir d'une *graine initiale*, par un processus de *mutation* qui les fait naturellement apparaître en sous-ensembles finis appelés *amas*. Lorsqu'une algèbre amassée est catégorifiée à l'aide d'une catégorie triangulée, ses variables d'amas et ses amas correspondent à certains objets rigides. Le premier chapitre regroupe des résultats de théorie des représentations concernant les algèbres d'endomorphismes de ces objets rigides. Nous tentons de classifier les algèbres d'endomorphismes d'objets rigides qui ne possèdent qu'un nombre fini de représentations indécomposables à isomorphisme près. Si deux objets rigides sont reliés par une mutation, nous comparons leurs carquois colorés, ainsi que les catégories des représentations de leurs algèbres d'endomorphismes.

Dans le chapitre 2, nous nous intéressons à la catégorification additive des algèbres amassées. Dans ce cadre, les variables d'amas s'obtiennent en appliquant un *caractère d'amas*, ou application de Caldero–Chapoton, aux objets rigides indécomposables. Après s'être intéressé aux algèbres amassées de rang infini, nous nous tournons vers le cas des algèbres amassées de type fini non simplement lacé : nous démontrons une formule de multiplication, inspirée des travaux de Philippe Caldero et Frédéric Chapoton, dans le cadre des algèbres associées aux matrices de Cartan symétrisables par Christof Geiß, Bernard Leclerc et Jan Schröer. Nous expliquons également comment utiliser le *type cone* de Peter McMullen afin d'obtenir toutes les réalisations polytopales des éventails de $\boldsymbol{g}$-vecteurs des algèbres amassées de type fini, en revisitant une approche due à Véronique Bazier-Matte, Guillaume Douville, Kaveh Mousavand, Hugh Thomas et Emine Yıldırım.

Le chapitre 3 concerne le τ-basculement des algèbres aimables, et ses liens avec la combinatoire algébrique. Après une introduction en douceur au cas des marches dans les grilles, nous expliquons comment certains résultats de combinatoire algébrique, dus à Thomas McConville et à Alexander Garver et Thomas McConville, peuvent se réinterpréter, puis se généraliser en utilisant la théorie des représentations des algèbres aimables. Ceci permet, en particulier, de montrer que la combinatoire des marches dans une grille, et celle des accordéons associés à une dissection de surface, peuvent être identifiées et sont une ombre combinatoire de la notion algébrique de τ-basculement.

Les catégories extriangulées, introduites en collaboration avec Hiroyuki Nakaoka, sont présentées dans le chapitre 4. Nous introduisons les premières notions liées à cette nouvelle structure catégorique avant de présenter des résultats de théorie d'Auslander–Reiten valables dans ce cadre. Nous en proposons ensuite des applications à la théorie des représentations des algèbres aimables et à la combinatoire des accordéons.

Un dernier chapitre déborde légèrement du cadre de la théorie des représentations pour s'intéresser à deux résultats d'algèbre homotopique :

- Une réinterprétation, à l'aide de structure de modèles, de résultats dus à Aslak B. Buan et Robert J. Marsh sur les algèbres d'endomorphismes d'objets rigides.

- Une extension de la correspondance de Hovey entre certaines paires de cotorsion et structures de modèles, du cas des catégories exactes à celui des catégories extriangulées.

Dans ce second point, nous montrons également que la catégorie homotopique d'une structure de modèles exacte est toujours triangulée. Pour les catégories exactes, ce résultat n'était connu que dans le cas des paires de cotorsion héréditaires.

Publications et Pré-publications

1. En collaboration avec Arnau Padrol, Vincent Pilaud et Pierre-Guy Plamondon: *Associahedra for finite type cluster algebras and minimal relations between g-vectors* [PPPP19].

2. En collaboration avec Vincent Pilaud et Pierre-Guy Plamondon: *Non-kissing and non-crossing complexes for locally gentle algebras* [PPP18].

3. En collaboration avec Osamu Iyama et Hiroyuki Nakaoka: *Auslander–Reiten theory in extriangulated categories* [INP18].

4. En collaboration avec Vincent Pilaud et Pierre-Guy Plamondon: *Non-kissing complexes and tau-tilting for gentle algebras*, a paraître dans Memoirs of the AMS [PPP17]

5. En collaboration avec Hiroyuki Nakaoka: *Extriangulated categories, Hovey twin cotorsion pairs and model structures*, Cahiers de Topologie et Géométrie Différentielle Catégoriques, Volume LX-2 (2019) [NP19].

6. En collaboration avec Aslak B. Buan et Idun Reiten: *Algebras of finite representation type arising from maximal rigid objects*, Journal of Algebra 446 (2016) [BPR16].

7. *From Triangulated categories to module categories via homotopical algebra*, (non soumis) [Pal14].

8. En collaboration avec Robert J. Marsh: *Nearly Morita equivalences and rigid objects*, Nagoya Math. J. Volume 225 (2017), 64–99 [MP17].

9. En collaboration avec Robert J. Marsh: *Coloured quivers for rigid objects and partial triangulations: The unpunctured case*, Proc. London Math. Soc. Vol 108, Number 2 (2014), 411–440 [MP14].

10. En collaboration avec Peter Jørgensen: *A Caldero–Chapoton map for infinite clusters*, Trans. Amer. Math. Soc. 365 (2013), 1125–1147 [JP13].

11. *Cluster characters II: A multiplication formula*, Proc. London Math. Soc., Vol 104, Number 1 (2012), 57–78 [Pal12].

12. *Grothendieck group and generalized mutation rule for 2-Calabi–Yau triangulated categories*, J. Pure Appl. Algebra 213 (2009) 1438–1449 [Pal09].

13. *Cluster characters for 2-Calabi–Yau triangulated categories,* Annales de l'Institut Fourier, 58 no. 6 (2008), 2221–2248 [Pal08].

Articles de conférence

- En collaboration avec Vincent Pilaud et Pierre-Guy Plamondon: *Non-kissing and non-crossing complexes for locally gentle algebras*, résumé étendu (FPSAC'19).

- En collaboration avec Vincent Pilaud et Pierre-Guy Plamondon: *Non-kissing complexes and tau-tilting for gentle algebras*, résumé étendu (FPSAC'18).

Poster

- Non-kissing vs non-crossing FPSAC'19 (Ljubljana)
  http://www.lamfa.u-picardie.fr/palu/36poster.pdf

Travaux en cours de rédaction mentionnés dans ce mémoire

- En collaboration avec Peter Jørgensen : *Tau-cotorsion pairs* [JP19].
- En collaboration avec Pierre-Guy Plamondon : *Cluster categories and a multiplication formula for skew-symmetrizable Cartan matrices* [PP].

Chapter 1

Endomorphism algebras of rigid objects in 2-Calabi–Yau triangulated categories.

This chapter gathers some results from three articles on the endomorphism algebras of rigid objects in 2-Calabi–Yau triangulated categories. For short, we call those algebras *endo-rigid algebras*.

In Section 1.1, our aim is to study the following problem: How can we describe the endomorphism algebra of some mutation of a rigid object directly from the endomorphism algebra of the initial rigid object?

This problem was solved by Aslak Buan and Hugh Thomas in [BT09] for the endomorphism algebras of d-cluster tilting objects in $(d+1)$-Calabi–Yau categories. In that case, it is not possible to recover the endomorphism algebra of some mutation $\mu_k T$ of a d-cluster tilting object T from the datum of the endomorphism algebra of T only. However, the authors enhance the Gabriel quiver of $\operatorname{End}(T)$ by adding new coloured arrows encoding "j-step" mutations for $j \leq d$. It is then possible to write an algorithm that recovers the coloured quiver of $\mu_k T$, and hence the Gabriel quiver of its endomorphism algebra, from the coloured quiver of T.

Even though this seems to fail for more general rigid objects, we mimic Buan–Thomas' strategy and give an algorithm solving the problem for rigid objects in cluster categories of type A_n (an adaptation to type D_n is possible, by using the punctured disk description of the cluster category). We only give a partial description of the coloured quiver of $\mu_k R$ starting from the coloured quiver of R for more general 2-Calabi–Yau categories.

As a tool for studying the case of cluster categories from unpunctured surfaces, we notice that their Iyama–Yoshino reductions [IY08] can be understood as cutting the surface along an arc. This is arguably the most interesting result of the paper.

A second question, solved in Section 1.2, is: How closely related is the category of modules over the endomorphism algebra of a rigid object to that of one of its mutations?

For any mutation of a cluster tilting object in a 2-Calabi–Yau category, the answer is roughly: they only differ by one simple module [BMR07]. The general case of rigid objects is slightly more involved, but gets a similar answer: after passing to some extension-closed subcategory of the module category, they only differ by one simple object.

In the last Section 1.3, we use Claire Amiot's classification [Ami07] of triangulated categories with finitely many isomorphism classes of indecomposable objects in order to list all endo-rigid algebras of finite representation type coming from maximal rigid objects in 2-Calabi–Yau triangulated categories.

Along the way, we recover Igor Burban, Osamu Iyama, Bernhard Keller and Idun Reiten's classification [BIKR08] of 2-Calabi–Yau categories with finitely many isoclasses of indecomposables having cluster tilting objects or non-zero maximal rigid objects respectively. Our classification differs from theirs by gathering those categories in very few families depending on one or two parameters, making the list easier to remember and highlighting the similarities between the categories arising from a same family.

I would also like to advertise the categories $D^{\mathrm{b}}(A_{3n})/\tau^n[1]$. Quite some work has been done on the cluster tube, *e.g.* [Vat11, Yan12, HJR14, ZZ14]. The various authors face technical difficulties, due to the fact that the cluster tube has infinitely many isomorphism classes of indecomposable objects, that it has non-trivial infinite radical, and that its maximal rigid objects are not cluster tilting. On the other hand, the categories $D^{\mathrm{b}}(A_{3n})/\tau^n[1]$ only have finitely many isomorphism classes of indecomposables, are the mesh categories of their Auslander–Reiten quivers, and have cluster tilting objects. I believe that everything that has been done, or is being done, with the cluster tube can be done more easily with the categories $D^{\mathrm{b}}(A_{3n})/\tau^n[1]$.

Endo-rigid algebras will make another appearance in the last chapter, Section 5.1: If R is some rigid object in a triangulated category $\mathscr{C}$, the module category of $\mathrm{End}_{\mathscr{C}}(R)$ is shown to be the homotopy category of some model structure on $\mathscr{C}$.

1.1 Coloured quivers for endo-rigid algebras.

In the article [MP14], we introduce mutations and coloured quivers for rigid objects in 2-Calabi–Yau triangulated categories. Our ideas are inspired from [BT09] which covers the case of d-cluster-tilting objects in $(d+1)$-Calabi–Yau triangulated categories. Even though the case of rigid objects seems too general to hope for a nice theory of mutations, we give a partial description of a mutation for coloured quivers in 2-Calabi–Yau categories and an algorithm for mutating coloured quivers in type A. In the case of cluster categories from surfaces, we relate mutation of rigid objects to flips of dissections. Our main tool is a combinatorial interpretation of Iyama–Yoshino reductions of cluster categories from surfaces: Reduction corresponds to cutting along arcs on the surface.

1.1.1 Mutations of rigid objects in 2-CY categories.

Let $\mathbb{K}$ be a field. Let $\mathscr{C}$ be a $\mathbb{K}$-linear, Hom-finite, Krull–Schmidt, 2-Calabi–Yau triangulated category. Let $R = R_1 \oplus \cdots \oplus R_m$ be a basic rigid object in $\mathscr{C}$ and let X be an indecomposable summand of R.

For $c \in \mathbb{Z}$, consider triangles

$$X^{(c)} \xrightarrow{f^c} B^{(c)} \xrightarrow{g^c} X^{(c+1)} \longrightarrow \Sigma X^{(c)}$$

where f^c is a minimal left $\operatorname{add} R/X$-approximation and g^c is a minimal right $\operatorname{add} R/X$-approximation and where $X^{(0)} = X$. These will be called the *exchange triangles* for X with respect to R. They can be constructed using induction on c. We will often write $\kappa_R^{(c)} X$ for $X^{(c)}$, and κ for $\kappa^{(1)}$; $\kappa_R X$ will be referred to as the *twist* of X with respect to R. Note that $\kappa\kappa^{(c)} = \kappa^{(c+1)} = \kappa^{(c)}\kappa$ for all c.

These exchange triangles lift the triangles $X^{(c)} \longrightarrow 0 \longrightarrow \Sigma_R X^{(c)} \longrightarrow \Sigma_R X^{(c)}$ in the Iyama–Yoshino reduction ${}^{\perp}(\Sigma R)/R$ canonically to $\mathscr{C}$. Therefore, $X^{(c)}$ is indecomposable, rigid and Ext-orthogonal to $\operatorname{add} R$ for all c. This justifies the following definition:

Definition 1.1.1. The *mutation* of R at R_k, where $k = 1, \ldots, m$, is the rigid object

$$\mu_{R_k} R = R/R_k \oplus \kappa_R R_k.$$

We note that our use of Iyama–Yoshino to define the mutation above is similar to that of [BØO11, Sect. 3] where cluster tilting objects are mutated at a non-indecomposable summand.

In [BT09], the authors associate coloured quivers to d-cluster–tilting objects in $(d+1)$-Calabi–Yau categories. Here we use the same definition to associate a coloured quiver to the rigid object R.

Definition 1.1.2. The *coloured quiver* $Q = Q_R$ associated with the rigid object R is defined as follows: The set of vertices is $Q_0 = \{1, \ldots, m\}$. The set $Q_1^{(c)}(i,j)$ of c-coloured arrows from i to j has cardinality given by the multiplicity of R_j in $B_i^{(c)}$, where

$$R_i^{(c)} \xrightarrow{f_i^c} B_i^{(c)} \xrightarrow{g_i^c} R_i^{(c+1)} \longrightarrow \Sigma R_i^{(c)}$$

are the exchange triangles as above for R_i with respect to R.

When the category $\mathscr{C}$ is a (generalised) cluster category, it is often the case that a sequence of exchange triangles is periodic. With each vertex k of Q is thus associated an integer d_k (possibly infinite): the periodicity of the sequence of exchange triangles for R_k. In order to avoid keeping infinitely many arrows starting at each vertex when not necessary, the colours of the arrows starting at a vertex k are considered as elements in $\mathbb{Z}/d_k$. Note that the periodicity depends on the starting vertex.

An algorithm giving the coloured quiver of a mutation $\mu_{R_k} R$ from the coloured quiver of R seems too much to hope for in this generality. However, a partial result holds:

Theorem 1.1.3. *Let $\mathscr{C}$ be a $\mathbb{K}$-linear, Hom-finite, Krull–Schmidt, 2-Calabi–Yau triangulated category. Let Q be the coloured quiver associated with a rigid object $R \in \mathscr{C}$ and let $\widetilde{Q}$ be the coloured quiver associated with $\mu_k R$, for some vertex k of Q. Denote the periodicity associated with vertex i of Q (resp. $\widetilde{Q}$) by d_i (resp. d'_i), and the number of arrows of Q (resp. $\widetilde{Q}$), from i to j and of colour c, by $q_{i,j}^{(c)}$ (resp. $\widetilde{q}_{i,j}^{(c)}$).*

(i) We have:

- $d_k = d'_k$ *and*
- *for any* $j \in Q_0$ *and any* $c \in \mathbb{Z}/d_k$, $\widetilde{q}^{(c)}_{k,j} = q^{(c+1)}_{k,j}$.

(ii) Let $i, j \in Q_0$ *be such that* $q^{(0)}_{k,j} = 0 = q^{(0)}_{k,i}$. *Then we have:*

- $d'_i = d_i$, $d'_j = d_j$;
- *for any* $c \in \mathbb{Z}/d_j$, $\widetilde{q}^{(c)}_{j,k} = q^{(c-1)}_{j,k}$;
- *for any* $c \in \mathbb{Z}/d_i$, $\widetilde{q}^{(c)}_{i,j} = q^{(c)}_{i,j}$.

Obviously, the proof heavily makes use of the triangulated structure of $\mathscr{C}$.

1.1.2 The case of cluster categories from surfaces.

Let $(\mathcal{S}, M)$ be a marked surface: $\mathcal{S}$ is an oriented surface with boundary and M is a finite set of marked points on the boundary of $\mathcal{S}$, with at least one marked point on each boundary component.

As explained and studied in [BZ11], the works [DWZ08, LF09, KY11, Ami09] allow to associate a 2-Calabi–Yau triangulated category to $(\mathcal{S}, M)$, called the cluster category of $(\mathcal{S}, M)$ and denoted by $\mathscr{C}_{(\mathcal{S},M)}$.

Remark 1.1.4. In fact, the category $\mathscr{C}_{(\mathcal{S},M)}$ depends on the choice of a triangulation of $(\mathcal{S}, M)$. As explained by Claire Amiot in the appendix of [CS17], the equivalences of categories induced by changing the choice of a triangulation are *not canonical* in general, leading to potential subtle mistakes. However, [CS17, Proposition A.2] shows that such problems can be avoided in the case of unpunctured surfaces.

Fix a marked surface $(\mathcal{S}, M)$ and let D be a dissection (i.e. a subset of a triangulation) on $(\mathcal{S}, M)$.

Definition 1.1.5. Let $\gamma \in \mathrm{D}$ and let α, β be the two arcs (either in D or boundary components) that follow γ in the clockwise ordering at each endpoint of γ (see Figure 1.1). The *flip* of D at γ is the dissection

$$\mu_\gamma \mathrm{D} = (\mathrm{D} \setminus \{\gamma\}) \cup \{\kappa_{\mathrm{D}}(\gamma)\}, \text{ where } \kappa_{\mathrm{D}}(\gamma) = \beta\gamma\alpha.$$

The arcs α, β are called the *supporting arcs* of the flip.

Let α and β be two arcs in $(\mathcal{S}, M)$, and let D be a dissection containing α. Let α_1, α_2 be the supporting arcs of the flip of D at α. For all $c \in \mathbb{Z}$, define the numbers $q^{(c)}_{\mathrm{D}}(\alpha, \beta)$ by:

$$\begin{aligned} q_{\mathrm{D}}(\alpha,\beta) &= \begin{cases} 2 & \text{if } \beta = \alpha_1 = \alpha_2 \\ 1 & \text{if } \beta \in \{\alpha_1, \alpha_2\} \text{ and } \alpha_1 \neq \alpha_2 \\ 0 & \text{otherwise,} \end{cases} \\ q^{(c)}_{\mathrm{D}}(\alpha,\beta) &= q_{\mathrm{D}}(\kappa^c_{\mathrm{D}}(\alpha), \beta). \end{aligned}$$

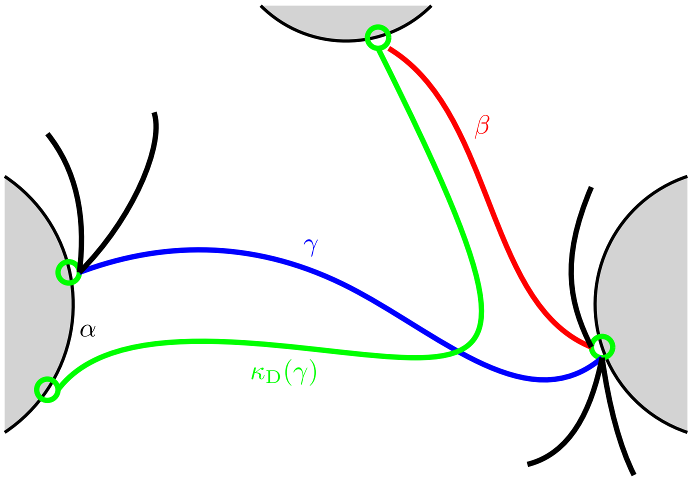


Figure 1.1: The flip of a dissection D at an arc γ: the arc γ (blue) is replaced by the arc $\kappa_{\mathrm{D}}(\gamma)$ (green).

Definition 1.1.6. Let $\mathrm{D} = \{\gamma_1, \dots, \gamma_m\}$ be a dissection of $(\mathcal{S}, M)$. The *coloured quiver* Q_{D} associated with D is defined as follows: The set of vertices is $Q_0 = \{1, \dots, m\}$. The set $Q_1^{(c)}(i,j)$ of c-coloured arrows from vertex i to vertex j has cardinality $q_{\mathrm{D}}^{(c)}(\gamma_i, \gamma_j)$.

We now fix a background triangulation of $(\mathcal{S}, M)$, and thus a cluster-tilting object $T \in \mathscr{C}_{(\mathcal{S},M)}$. These data give a bijection between arcs γ on $(\mathcal{S}, M)$ and indecomposable objects X_γ in $\mathscr{C}_{(\mathcal{S},M)}$ such that $\mathscr{C}_{(\mathcal{S},M)}(T, X_\gamma)$ is a string module over the gentle algebra $\mathrm{End}_{\mathscr{C}_{(\mathcal{S},M)}}(T)$. Recall from [BZ11] that, under this bijection, two arcs α, β cross if and only if $\mathscr{C}_{(\mathcal{S},M)}(X_\alpha, \Sigma X_\beta) \neq 0$.

Proposition 1.1.7. *Let R be the rigid object in $\mathscr{C}_{(\mathcal{S},M)}$ associated with a dissection* D. *Let $(\mathcal{S}, M)/\mathrm{D}$ be the marked surface obtained from $(\mathcal{S}, M)$ by cutting along the arcs in* D. *Then the Iyama–Yoshino reduction ${}^{\perp}(\Sigma R)/(R)$ of $\mathscr{C}_{(\mathcal{S},M)}$ is equivalent to the cluster category $\mathscr{C}_{(\mathcal{S},M)/\mathrm{D}}$.*

Proposition 1.1.8. *Let R be the rigid object in $\mathscr{C}_{(\mathcal{S},M)}$ associated with a dissection* D. *Let R_k be an indecomposable summand of R, and let γ_k be the corresponding arc of* D. *For any $\alpha \in \mathrm{D}$, we have:*

$$\kappa_R X_\alpha \cong X_{\kappa_{\mathrm{D}}(\alpha)} \text{ and } \mu_{R_k} R \cong X_{\mu_k \mathrm{D}}.$$

Theorem 1.1.9. *Let R be the rigid object in $\mathscr{C}_{(\mathcal{S},M)}$ associated with some dissection* D. *Then the coloured quivers Q_{D} and Q_R coincide.*

As a corollary of Theorem 1.1.9, we give an explicit algorithm for computing the coloured quiver of a mutation $\mu_{R_k} R$ of a rigid object R in a cluster category of type A, directly from the coloured quiver of R.

1.2 Pseudo-Morita equivalences of endorigid algebras.

In the article [MP17] in collaboration with Robert J. Marsh, we compare the module categories over the endomorphism algebras of two rigid objects related by a single mutation. We work in the setup of Krull–Schmidt, Hom-finite, triangulated categories. Further generalisation to cotorsion pairs has appeared in [Nak16].

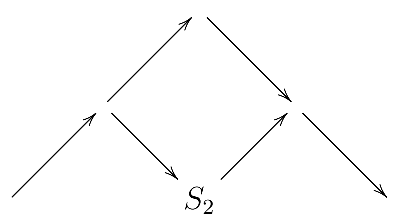
1.2.1 Motivating example.

Let Q be a linear orientation of the Dynkin diagram of type A_3. The Auslander–Reiten quiver of the acyclic cluster category $\mathscr{C}_Q$, defined in [BMR+06], is as follows:

$$\begin{tikzcd}
 & & T_3 \arrow[dr] & & \Sigma T_1 \arrow[dr] & & T_1 \arrow[dr] & & \Sigma T_2^* \\
 & T_2 \arrow[ur] \arrow[dr] & & {} \arrow[ur] \arrow[dr] & & \Sigma T_2 \arrow[ur] \arrow[dr] & & T_2 \arrow[ur] \arrow[dr] & \\
T_1 \arrow[ur] & & \Sigma T_2^* \arrow[ur] & & T_2^* \arrow[ur] & & \Sigma T_3 \arrow[ur] & & T_3
\end{tikzcd}$$

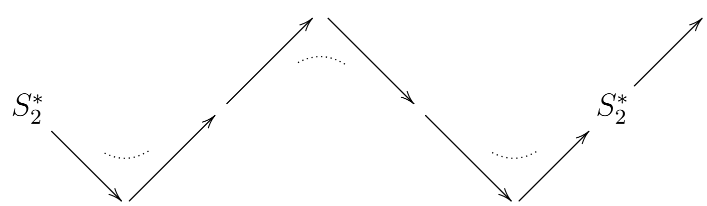
The object $T = T_1 \oplus T_2 \oplus T_3$ is cluster-tilting. Its mutation at T_2 is the cluster-tilting object $T' = T_1 \oplus T_2^* \oplus T_3$. We write Γ for the cluster-tilted algebra $\mathrm{End}_{\mathscr{C}}(T)^{\mathrm{op}}$ and Γ' for $\mathrm{End}_{\mathscr{C}}(T')^{\mathrm{op}}$. Then the two algebras Γ and Γ' are related as follows.

On the one hand, the functor $\mathscr{C}(T, -)$ induces an equivalence of categories $\mathscr{C}/(\Sigma T) \simeq \mathrm{mod}\,\Gamma$, where $\mathrm{mod}\,\Gamma$ is the category of finitely generated left modules, and the Auslander–Reiten quiver of $\mathrm{mod}\,\Gamma$ is thus:



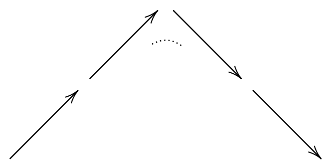
where $S_2 = \mathscr{C}(T, \Sigma T_2^*)$ is the simple top of the projective indecomposable $\mathscr{C}(T, T_2)$.

On the other hand, the functor $\mathscr{C}(T', -)$ induces an equivalence of categories $\mathscr{C}/(\Sigma T') \simeq \mathrm{mod}\,\Gamma'$ and the Auslander–Reiten quiver of $\mathrm{mod}\,\Gamma'$ is thus:



where $S_2^* = \mathscr{C}(T', \Sigma T_2)$ is the simple top of the projective indecomposable $\mathscr{C}(T', T_2^*)$, where the two arrows starting at S_2^* are identified, and where dots indicate zero relations.

The two Auslander–Reiten quivers are not isomorphic, therefore Γ and Γ' are not Morita equivalent. But they are not very far from being so: The difference in the Auslander–Reiten quivers comes from the simples S_2 and S_2^*. The common Auslander-Reiten quiver of the categories $\operatorname{mod}\Gamma/(\operatorname{add} S_2)$ and $\operatorname{mod}\Gamma/(\operatorname{add} S_2^*)$ is thus:



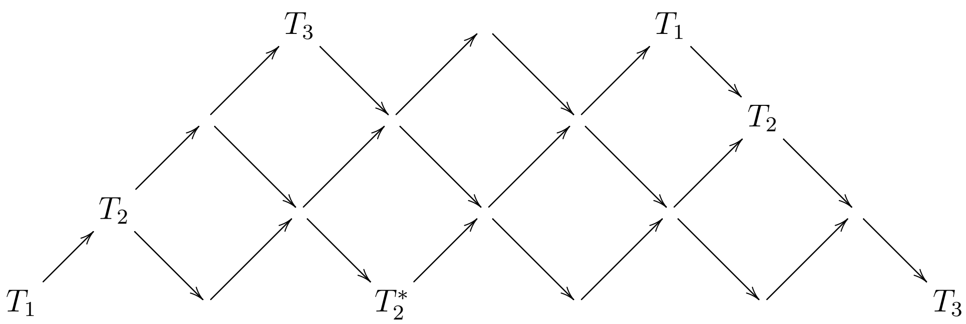
This phenomenon, proved in [BMR07], has been called "nearly Morita equivalence" by Claus M. Ringel. Let us state the precise result.

Let Q be an acyclic quiver, and let T be a basic cluster-tilting object in the cluster category $\mathscr{C}_Q$. Let $T' = T/T_k \oplus T_k^*$ be the mutation of T at an indecomposable summand T_k; then T' is also a cluster-tilting object. Let Γ (respectively, Γ') be the cluster-tilted algebra $\operatorname{End}_{\mathscr{C}_Q}(T)^{\mathrm{op}}$ (respectively, $\operatorname{End}_{\mathscr{C}_Q}(T')^{\mathrm{op}}$) and S_k (respectively, S_k^*) be the simple top of the projective indecomposable Γ-module $\mathscr{C}_Q(T, T_k)$ (respectively, the simple top of the Γ'-module $\mathscr{C}_Q(T', T_k^*)$).

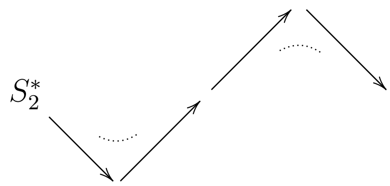
Then, by [BMR07, Theorem B.], the categories $\operatorname{mod}\Gamma/\operatorname{add} S_k$ and $\operatorname{mod}\Gamma'/\operatorname{add} S_k^*$ are equivalent. By [Yan12, Corollary 4.3], nearly-Morita equivalence, in the more general setup of simple, 2-periodic mutations of rigid objects (or rigid, Krull–Schmidt subcategories) in any triangulated category, follows from [Pla11b, Proposition 2.7].

Our main aim is to prove an analogous result for any mutation of (non-maximal) rigid objects. Before explaining our results, let us have a look at an example which shows that one cannot expect these mutations to induce a nearly-Morita equivalence in general.

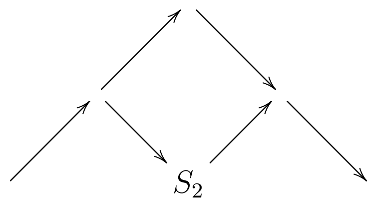
Let $T = T_1 \oplus T_2 \oplus T_3$ be the rigid object of the acyclic cluster category $\mathscr{C} = \mathscr{C}_{A_4}$ given by:



and let $T' = T_1 \oplus T_2^* \oplus T_3$ be the rigid object obtained by mutating T at the summand T_2. This means that ΣT_2^* is the cone of a minimal right $\operatorname{add} T/T_2$-approximation of T_2. In the example, there is a triangle $T_2^* \to T_1 \to T_2 \to \Sigma T_2^*$. Let Λ (respectively, Λ') be the algebra $\operatorname{End}_{\mathscr{C}}(T)^{\mathrm{op}}$ (respectively, $\operatorname{End}_{\mathscr{C}}(T')^{\mathrm{op}}$). Using results in [BM13, BM12] (see also [KR07]), we can easily compute the AR quivers of $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Lambda'$:

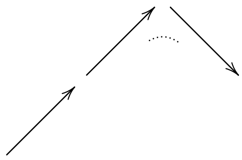


$\operatorname{mod}\Lambda'$

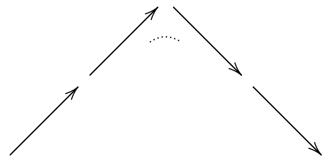


$\operatorname{mod}\Lambda$

On factoring out by S_2 (respectively, S_2^*), we obtain the following Auslander-Reiten quivers:



$\operatorname{mod}\Lambda'/\operatorname{add} S_2^*$



$\operatorname{mod}\Lambda/\operatorname{add} S_2$

Therefore, the algebras Λ and Λ' are not nearly-Morita equivalent. However, these algebras are not very far from being so: their Auslander–Reiten quivers differ by only one arrow. The corresponding morphism can be characterised in $\operatorname{mod}\Lambda$ as being surjective with kernel in the subcategory $\operatorname{add} S_2$. We are thus tempted to invert all such morphisms. This is precisely what we will do in Section 1.2.3. Because Nearly-Morita equivalences do not involve localisations, but rather ideal quotients, we first give, in Section 1.2.2 a different version of the statement, that more closely resembles nearly-Morita equivalence, and which we call:

1.2.2 Pseudo-Morita equivalence.

We fix a field $\mathbb{K}$, and a Krull–Schmidt, $\mathbb{K}$-linear, Hom-finite, triangulated category $\mathscr{C}$, with suspension functor Σ. Let $T \in \mathscr{C}$ be a basic rigid object. Let R be a (non-necessarily indecomposable) direct summand of T and write $T = \overline{T} \oplus R$. Let T' be the object obtained from T by replacing R by the negative shift R^* of the cone of a minimal right $\operatorname{add}\overline{T}$-approximation of R. We thus have a triangle $R^* \to B \to R \to \Sigma R^*$, with $B \in \operatorname{add}\overline{T}$, $B \to R$ a minimal right $\operatorname{add}\overline{T}$-approximation, and $T' = \overline{T} \oplus R^*$. By [BMR+06, Lemma 6.7], $\Sigma R^* \in \overline{T}^{\perp}$. We assume moreover that T' is rigid. More precisely, we assume that R^* belongs to ${}^{\perp}\Sigma\overline{T}$ (then R^* is automatically rigid). This holds for example when $\mathscr{C}$ is 2-Calabi–Yau. By [IY08, Proposition 2.6(1)] and [BMR+06, Lemma 6.5], R and R^* are basic and have the same number of indecomposable direct summands.

Proposition 1.2.1. *The inclusion functor $I : T * \Sigma\overline{T} \to \mathscr{C}$ induces an adjunction*

$$I \dashv R : (T * \Sigma\overline{T})/(\Sigma T') \rightleftarrows \mathscr{C}/(\overline{T}^{\perp}).$$

Proposition 1.2.2. *Assume that $\mathscr{C}$ has a Serre functor $\mathbb{S}$. Then the inclusion functor $J : (\Sigma^{-1}\mathbb{S}\overline{T}) * \mathbb{S}T' \to \mathscr{C}$ induces an adjunction*

$$L \dashv J : \left((\Sigma^{-1}\mathbb{S}\overline{T}) * \mathbb{S}T'\right) / (\Sigma^{-1}\mathbb{S}T) \rightleftarrows \mathscr{C}/(\overline{T}^{\perp}).$$

The key to proving those two propositions is the lemma below, inspired from [BM13, Lemma 3.3]:

Lemma 1.2.3. *(a) For each object $X \in \mathscr{C}$, there is a triangle $Y \to R_0X \to X \xrightarrow{g} \Sigma Y$, with $R_0X \in T * \Sigma\overline{T}$, $Y \in \overline{T}^{\perp}$ and $g \in (T^{\perp})$.*

(b) The subcategory $T * \Sigma\overline{T}$ is contravariantly finite in $\mathscr{C}$.

Assuming that $\mathscr{C}$ has a Serre functor $\mathbb{S}$, we thus have a pair of adjoint functors (G, H), where $G = LI$ and $H = RJ$. We note that, since I, J, L and R are additive, so are G and H.

$$\begin{tikzcd}
& \mathscr{C}/(\overline{T}^{\perp}) \arrow[dl, shift left, "R"] \arrow[dr, shift left, "L"] & \\
T * \Sigma\overline{T}/(\Sigma T') \arrow[ur, shift left, "I"] \arrow[rr, shift left, "G"] & & \left((\Sigma^{-1}\mathbb{S}\overline{T}) * \mathbb{S}T'\right) / (\Sigma^{-1}\mathbb{S}T) \arrow[ul, shift left, "J"] \arrow[ll, shift left, "H"]
\end{tikzcd}$$

Theorem 1.2.4. *Assume that $\mathscr{C}$ has a Serre functor $\mathbb{S}$. Then the functors G and H are quasi-inverse equivalences of categories. In particular, the categories $T * \Sigma\overline{T}/(\Sigma T')$ and $T * \Sigma\overline{T}/(T)$ are equivalent.*

We now want to interpret the subcategories $T*\Sigma\overline{T}$ and $(\Sigma^{-1}\mathbb{S}\overline{T})*\mathbb{S}T'$ in representation-theoretic terms.

We write $\overline{T} = T_1 \oplus \cdots \oplus T_n$ and $R = T_{n+1} \oplus \cdots \oplus T_m$, where the T_i are indecomposable. We have $T' = \overline{T} \oplus R^* = T'_1 \oplus \cdots \oplus T'_m$ where $T'_i = T_i$ if $i \leq n$. Define Λ, to be the endomorphism algebra $\mathrm{End}_{\mathscr{C}}(T)^{\mathrm{op}}$. The algebra Λ' is defined similarly to be $\mathrm{End}_{\mathscr{C}}(T')^{\mathrm{op}}$.

Let S_j be the simple top of the indecomposable projective Λ-module $\mathscr{C}(T, T_j)$, and let S'_j be the simple socle of the indecomposable injective Λ'-module $D\mathscr{C}(\Sigma T'_j, \Sigma T')$. We consider the exact categories $\mathscr{E}$ and $\mathscr{E}'$ defined as follows. The category $\mathscr{E}$ is the full subcategory of $\mathrm{mod}\,\Lambda$, whose objects are those M that satisfy $\mathrm{Ext}^1_{\Lambda}(M, S_j) = 0$ for all $j > n$. Similarly, the category $\mathscr{E}'$ is the full subcategory of $\mathrm{mod}\,\Lambda'$ whose objects N satisfy $\mathrm{Ext}^1_{\Lambda'}(S'_j, N) = 0$ for all $j > n$.

The reason for considering those specific exact subcategories is:

Lemma 1.2.5. *(a) The functor $\mathscr{C}(T, -)$ induces a fully faithful functor*

$$T * \Sigma\overline{T}/(\Sigma\overline{T}) \longrightarrow \mathrm{mod}\,\Lambda.$$

Its essential image is $\mathscr{E}$.

(b) Dually, the functor $D\mathscr{C}(-,\Sigma T')$ induces a fully faithful functor

$$T * \Sigma\overline{T}/(\overline{T}) \longrightarrow \operatorname{mod}\Lambda'.$$

Its essential image is $\mathscr{E}'$.

We obtain the following reformulation of Theorem 1.2.4, which is now similar to, but weaker than, nearly-Morita equivalence [BMR07, Theorem B.]:

Theorem 1.2.6. *Suppose that $\mathscr{C}$ has a Serre functor. Then there is an equivalence of categories:*

$$\mathscr{E}/\operatorname{add}\mathscr{C}(T,\Sigma R^*) \simeq \mathscr{E}'/\operatorname{add}D\mathscr{C}(R,\Sigma T').$$

Remark 1.2.7. If moreover, R is indecomposable, then $\mathscr{C}(T,\Sigma R^*)$ might not be a simple module. However, it is simple when seen as an object in $\mathscr{E}$.

1.2.3 Localisation.

We do not assume in this section that $\mathscr{C}$ has a Serre functor, except in Corollary 1.2.10.

Recall that, by [KR07, KZ08, IY08, BM13], the functor $\mathscr{C}(T,-)$ induces an equivalence of categories from $(T*\Sigma T)/\Sigma T$ to $\operatorname{mod}\Lambda$. In particular, it is dense and full when restricted to $T*\Sigma T$.

Let $\mathcal{B}$ be the essential image of $\mathscr{C}(T,-) : \overline{T}^{\perp} \to \operatorname{mod}\Lambda$. Let $\mathcal{S}_{\mathcal{B},0}$ be the class of all epimorphisms $f \in \operatorname{mod}\Lambda$ whose kernel belongs to $\mathcal{B}$. Dually, we let $\mathcal{B}'$ be the essential image of $D\mathscr{C}(-,\Sigma T') : {}^{\perp}\Sigma\overline{T} \to \operatorname{mod}\Lambda'$, and set $\mathcal{S}_{0,\mathcal{B}'}$ to be the class of all monomorphisms $g \in \operatorname{mod}\Lambda'$ whose cokernel belongs to $\mathcal{B}'$.

Let F be the composition of the fully faithful functor

$$(T*\Sigma\overline{T})/\Sigma\overline{T} \to (T*\Sigma T)/\Sigma T \to \operatorname{mod}\Lambda$$

and the localisation functor $\operatorname{mod}\Lambda \xrightarrow{L_{\mathcal{S}_{\mathcal{B},0}}} (\operatorname{mod}\Lambda)[(\mathcal{S}_{\mathcal{B},0})^{-1}]$. Then, since $\mathscr{C}(T,\Sigma R^*)$ belongs to $\mathcal{B}$, we have that $F(\Sigma R^*) \simeq 0$ in $(\operatorname{mod}\Lambda)[(\mathcal{S}_{\mathcal{B},0})^{-1}]$. Hence, F induces a functor $\overline{F}$ as in the following diagram:

$$\begin{tikzcd}
T*\Sigma\overline{T}/(\Sigma\overline{T}) \arrow[r, hook] \arrow[d] \arrow[drr, "F"] & (T*\Sigma T)/(\Sigma T) \arrow[r, "\simeq"] & \operatorname{mod}\Lambda \arrow[d, "L_{\mathcal{S}_{\mathcal{B},0}}"] \\
T*\Sigma\overline{T}/(\Sigma T') \arrow[rr, dashed, "\overline{F}"] & & (\operatorname{mod}\Lambda)[(\mathcal{S}_{\mathcal{B},0})^{-1}].
\end{tikzcd} \tag{1.1}$$

The ideal subquotients appearing in Theorem 1.2.6 can alternatively be described as localisations of the whole module category:

Theorem 1.2.8. *The functor $\overline{F} : T*\Sigma\overline{T}/(\Sigma T') \longrightarrow (\operatorname{mod}\Lambda)[(\mathcal{S}_{\mathcal{B},0})^{-1}]$ is an equivalence of categories. Dually, there is an equivalence $T*\Sigma\overline{T}/(T) \longrightarrow (\operatorname{mod}\Lambda')[(\mathcal{S}_{0,\mathcal{B}'})^{-1}]$.*

Corollary 1.2.9. *There is an equivalence of categories*

$$(\operatorname{mod}\Lambda)[(\mathcal{S}_{\mathcal{B},0})^{-1}] \simeq \mathscr{E}/\operatorname{add}\mathscr{C}(T,\Sigma R^*)$$

and, dually, an equivalence of categories

$$(\operatorname{mod}\Lambda')[(\mathcal{S}_{0,\mathcal{B}'})^{-1}] \simeq \mathscr{E}'/\operatorname{add}D\mathscr{C}(R,\Sigma T').$$

Corollary 1.2.10. *If the category $\mathscr{C}$ admits a Serre functor, then there is an equivalence of categories:*

$$(\operatorname{mod}\Lambda)[(\mathcal{S}_{\mathcal{B},0})^{-1}] \simeq (\operatorname{mod}\Lambda')[(\mathcal{S}_{0,\mathcal{B}'})^{-1}].$$

Insired by [BM13], we now describe $(\operatorname{mod}\Lambda)[(\mathcal{S}_{\mathcal{B},0})^{-1}]$ as a localisation of $\mathscr{C}$. This is in fact used in the proof of Theorem 1.2.8.

Notation 1.2.11. We define two classes of morphisms $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ as follows. A morphism f which is part of a triangle $Z \xrightarrow{g} X \xrightarrow{f} Y \xrightarrow{h} \Sigma Z$, belongs to the class $\mathcal{S}$ if Z belongs to $\overline{T}^{\perp}$ and h factors through $T^{\perp}$. It belongs to the class $\widetilde{\mathcal{S}}$ if g factors through $\overline{T}^{\perp}$ and h factors through $T^{\perp}$. We note that those conditions do not depend on the choice of a triangle containing f.

Remark 1.2.12. The right approximation $R_0X \to X$ constructed in Lemma 1.2.3 belongs to the class $\mathcal{S}$.

Theorem 1.2.13. *The functor $\mathscr{C}(T,-)$ induces an equivalence of categories:*

$$\mathscr{C}(T,-) : \mathscr{C}[\widetilde{\mathcal{S}}^{-1}] \xrightarrow{\simeq} (\operatorname{mod}\Lambda)[(\mathcal{S}_{\mathcal{B},0})^{-1}].$$

By combining Theorems 1.2.6 and 1.2.13 we can prove:

Theorem 1.2.14. *The localisation functor $\mathscr{C} \to \mathscr{C}[\widetilde{\mathcal{S}}^{-1}]$ induces an equivalence of categories:*

$$\mathscr{C}[\mathcal{S}^{-1}] \xrightarrow{\simeq} \mathscr{C}[\widetilde{\mathcal{S}}^{-1}].$$

1.3 Endorigid algebras of finite representation type.

In this section, we recall the results from [BPR16] in collaboration with Aslak B. Buan and Idun Reiten. We also include the list of all algebraic, d-Calabi–Yau triangulated categories with finitely many isomorphism classes of indecomposable objects, that can easily be obtained from [Ami07].

Unless stated otherwise, $\mathbb{K}$ will be an algebraically closed field of characteristic zero. We write Σ for the shift functor in any orbit category, and [1] for the shift in any derived category. We will use the following notation:

$$\mathcal{A}_{n,t} = D^{\mathrm{b}}\left(\mathbb{K}\mathrm{A}_{(2t+1)(n+1)-3}\right)/\tau^{t(n+1)-1}[1]$$

$$\mathcal{D}_{n,t} = D^{\mathrm{b}}\left(\mathbb{K}\mathrm{D}_{2t(n+1)}\right)/\tau^{n+1}\varphi^{n},$$

where φ is induced by an automorphism of order 2 of $\mathrm{D}_{2t(n+1)}$. The orbit categories that we consider are triangulated, by [Kel05, Theorem 1].

1.3.1 Zoology.

In [Ami07, Theorem 7.2], Claire Amiot classified all standard algebraic triangulated categories with finitely many isomorphism classes of indecomposable objects. By using geometric descriptions in type A [CCS06] and in type D [Sch14], and direct computations in type E, Igor Burban, Osamu Iyama, Bernhard Keller and Idun Reiten extracted from Claire Amiot's list all 2-Calabi–Yau triangulated categories with cluster tilting objects, and with non-zero maximal rigid objects (see the appendix of [BIKR08]). In this section, we give a restatement of the results in the appendix of [BIKR08]. We note two changes from their lists:

(L1) The orbit category $D^{\mathrm{b}}(\mathbb{K}\mathrm{E}_8)/\tau^4$ has cluster tilting objects (this case was first noticed by Sefi Ladkani in [Lad14, Section 1.4]);

(L2) The orbit category $D^{\mathrm{b}}(\mathbb{K}\mathrm{D}_4)/\tau^2\varphi$, where φ is induced by an automorphism of D_4 of order 2, has non-zero maximal rigid objects which are not cluster tilting.

Proposition 1.3.1 (Amiot ; Burban–Iyama–Keller–Reiten)**.** *The standard, 2-Calabi–Yau, triangulated categories with finitely many isomorphism classes of indecomposable objects and with cluster tilting objects are exactly the cluster categories of Dynkin types* A, D *or* E *and the orbit categories:*

- (Type A) $D^b(\mathbb{K}\mathrm{A}_{3n})/\tau^n[1]$, *where* $n \geq 1$*;*
- (Type D) $D^b(\mathbb{K}\mathrm{D}_{kn})/(\tau\varphi)^n$, *where* $n \geq 1$, $k > 1$, $kn \geq 4$ *and* φ *is induced by an automorphism of* D_{kn} *of order 2;*
- (Type E) $D^b(\mathbb{K}\mathrm{E}_8)/\tau^4$ *and* $D^b(\mathbb{K}\mathrm{E}_8)/\tau^8$.

Remark 1.3.2. The first two families can presumably serve as categorifications of cluster algebras of type B, C. As shown by Sefi Ladkani [Lad14], the category $D^{\mathrm{b}}(\mathbb{K}\mathrm{E}_8)/\tau^4$ categorifies cluster algebras of type G_2. Surprisingly, the category $D^{\mathrm{b}}(\mathbb{K}\mathrm{E}_8)/\tau^8$ only has 24 indecomposable rigid objects and thus does not categorify cluster algebras of type F_4 which have 28 cluster variables. It might be interesting to investigate the image of $D^{\mathrm{b}}(\mathbb{K}\mathrm{E}_8)/\tau^8$ under the Caldero–Chapoton map.

Proposition 1.3.3 (Amiot ; Burban–Iyama–Keller–Reiten)**.** *The standard, 2-Calabi–Yau, triangulated categories with finitely many isomorphism classes of indecomposable objects and with non-zero maximal rigid objects which are not cluster tilting are exactly the orbit categories:*

- (Type A) $D^b\big(\mathbb{K}\mathrm{A}_{(2t+1)(n+1)-3}\big)/\tau^{t(n+1)-1}[1]$, *where* $n \geq 1$ *and* $t > 1$*;*
- (Type D) $D^b\big(\mathbb{K}\mathrm{D}_{2t(n+1)}\big)/\tau^{n+1}\varphi^n$, *where* $n, t \geq 1$, *and where* φ *is induced by an automorphism of* $\mathrm{D}_{2t(n+1)}$ *of order 2 ;*
- (Type E) $D^b(\mathbb{K}\mathrm{E}_7)/\tau^2$ *and* $D^b(\mathbb{K}\mathrm{E}_7)/\tau^5$.

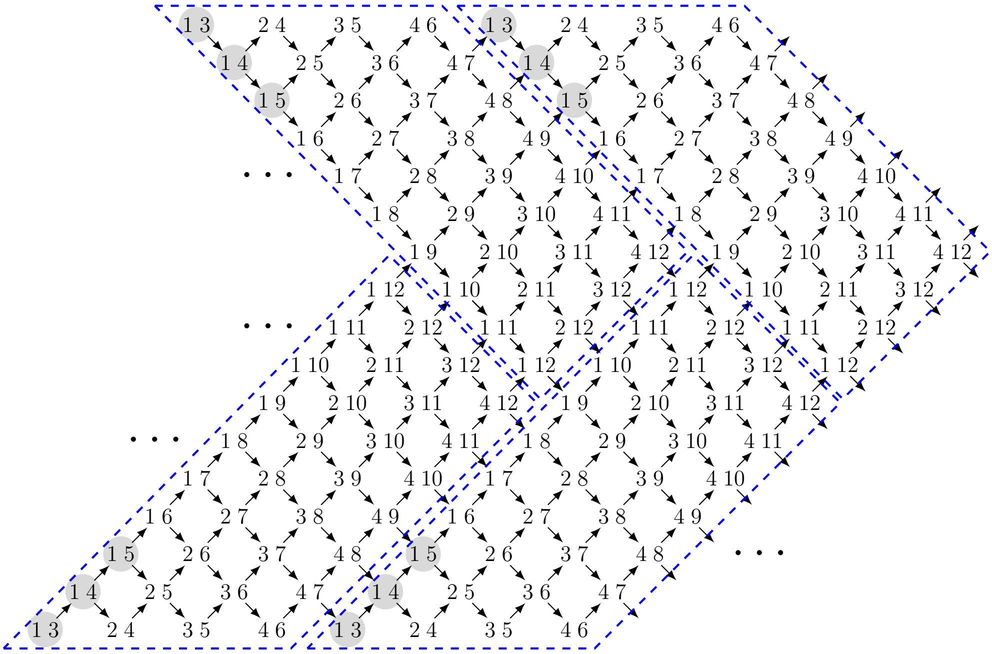


Figure 1.2: A bijection between $\frac{2\pi}{5}$-periodic collections of arcs of the heptakaidecagon and isomorphism classes of basic objects in $\mathcal{A}_{3,2}$. The maximal rigid object used in Table 2 is highlighted in grey.

The proofs of these statements make use of combinatorial descriptions, involving periodic collections of arcs, of the aforementioned categories. We illustrate this method with periodic collections of arcs in type A. Figure 1.2 shows the Auslander–Reiten quiver of the triangulated category $\mathcal{A}_{3,2}$, with a fundamental domain circled in dashed blue lines, where all indecomposables have been replaced by the corresponding diagonals of the heptakaidecagon. The periodic collection of diagonals of the icosikaipentagon associated with the maximal rigid object of Table 2 in the category $\mathcal{A}_{4,2}$ is drawn in Figure 1.3. Finally, an example of a periodic collection of diagonals corresponding to a non-rigid indecomposable object in $\mathcal{A}_{4,2}$ can be found in Figure 1.4. We make use of the same combinatorics in order to explicit a maximal rigid object in each of the families of 2-Calabi–Yau triangulated category of Propositions 1.3.1 and 1.3.3, and to describe their endomorphism algebras. We note that they are all Jacobian algebras of quivers with potentials (necessary care being taken depending on the characteristic of the ground field) except for one that is discussed in the next section. This section ends with the list of all algebraic, d-Calabi–Yau triangulated categories with finitely many isomorphism classes of indecomposable objects, extracted from Claire Amiot's list. It is out of reach to classify all those that have cluster-tilting or maximal rigid objects, since this would require decomposing integers into products of primes. However, for each small value of d, it is possible to mimic the method above and to give the equivalent of Tables 1 and 2.

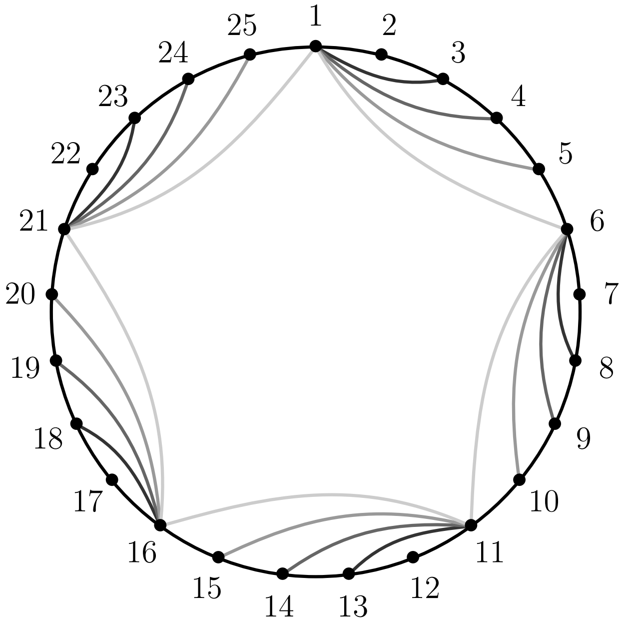


Figure 1.3: A collection of arcs of the icosikaipentagon corresponding to a maximal rigid object in $\mathcal{A}_{4,2}$. Each shade of grey appears five times and corresponds to one indecomposable object.

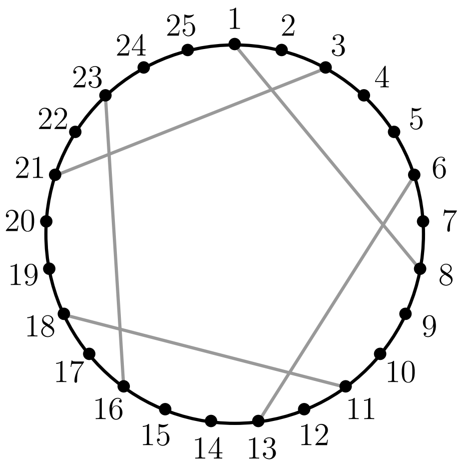


Figure 1.4: A collection of arcs of the icosikaipentagon corresponding to a non-rigid indecomposable object of $\mathcal{A}_{4,2}$.

Table 1: Orbit categories with cluster tilting objects, which are not acyclic cluster categories.

Orbit category	Indecomposables	Rank	Indec. rigids	Quiver	Relations
$D^{\mathrm{b}}(\mathbb{K}\mathrm{A}_{3n})/\tau^n[1]$	$\frac{3n(n+1)}{2}$	n	$n(n+1)$	$1 \to 2 \to 3 \cdots n-1 \to n \circlearrowright \alpha$	α^2
$D^{\mathrm{b}}(\mathbb{K}\mathrm{D}_{kn})/\tau^n\varphi^n$, $kn \geq 4$, $k > 1$	kn^2	n	$n(n+1)$	$1 \to 2 \to 3 \cdots n-1 \underset{b}{\overset{a}{\rightleftarrows}} n \circlearrowright \alpha$	$\alpha^{k-1} - ab$, αa, $b\alpha$
$D^{\mathrm{b}}(\mathbb{K}\mathrm{E}_8)/\tau^4$	32	2	8	$1 \to 2 \circlearrowright \alpha$	α^3
$D^{\mathrm{b}}(\mathbb{K}\mathrm{E}_8)/\tau^8$	64	4	24	$1 \to 2 \underset{b}{\overset{a}{\rightleftarrows}} 3 \to 4$	aba, bab

Table 2: Orbit categories with non-cluster tilting, maximal rigid objects.

Orbit category	Indecomposables	Rank	Indec. rigids	Quiver	Relations
$D^{\mathrm{b}}\big(\mathbb{K}\mathrm{A}_{(2t+1)(n+1)-3}\big)/\tau^{k(n+1)-1}[1]$ $t > 1$	$\frac{1}{2}[(2t+1)(n+1)-3](n+1)$	n	$n(n+1)$	$1 \to 2 \to 3 \cdots n-1 \to n \circlearrowright \alpha$	α^2
$D^{\mathrm{b}}\big(\mathbb{K}\mathrm{D}_{2t(n+1)}\big)/\tau^{n+1}\varphi^n$	$2t(n+1)^2$	n	$n(n+1)$	$1 \to 2 \to 3 \cdots n-1 \to n \circlearrowright \alpha$	α^2
$D^{\mathrm{b}}(\mathbb{K}\mathrm{E}_7)/\tau^2$	14	1	2	$1 \circlearrowright \alpha$	α^3
$D^{\mathrm{b}}(\mathbb{K}\mathrm{E}_7)/\tau^5$	35	2	5	$\alpha \circlearrowright 1 \xrightarrow{\beta} 2 \circlearrowright \gamma$	$\beta\alpha - \gamma\beta$, α^2, γ^2

Dynkin type	Restriction on d	Auto-equivalence	Restrictions on k
A_n, n odd	d even	$\tau^{k+\frac{n+1}{2}}[1]$	$k \mid \frac{(d-1)n+(d+1)}{2}$, and $\frac{(d-1)n+(d+1)}{2k}$ odd
	d odd	τ^k	$k \mid \frac{(d-1)n+(d+1)}{2}$
A_n, n even	any d	$\left(\tau^{\frac{n}{2}}[1]\right)^k$	$k \mid (d-1)n+(d+1)$
D_n, n odd	d even	$\tau^k\varphi$	$k \mid (d-1)n-(d-2)$
	d odd	τ^k	$k \mid (d-1)n$
		$\tau^k\varphi$	$k \mid (d-1)n$ and $\frac{(d-1)n}{k}$ even
D_4	any d	$\tau^k\sigma$, $\sigma \in \mathfrak{S}_3$	$k \mid 3d-2$ and $\sigma^{\frac{3d-2}{k}} = 1$
D_n, $n > 4$ even	d even	τ^k	$k \mid (d-1)n-(d-2)$
		$\tau^k\varphi$	$k \mid (d-1)n-(d-2)$ and $\frac{(d-1)n-(d-2)}{k}$ even
	d odd	τ^k	$k \mid (d-1)n-(d-2)$
E_6	d even	$\tau^k\varphi$	$k \mid 1+6(d-1)$ and $\frac{1+6(d-1)}{k}$ odd
	d odd	τ^k	$k \mid 1+6(d-1)$
		$\tau^k\varphi$	$k \mid 1+6(d-1)$ and $\frac{1+6(d-1)}{k}$ even
E_7	any d	τ^k	$k \mid 1+9(d-1)$
E_8	any d	τ^k	$k \mid 1+15(d-1)$

The d-Calabi–Yau triangulated orbit categories with finitely many isoclasses of indecomposables.

Here, φ is induced by the only non-trivial automorphism of the Dynkin diagram.

1.3.2 Classification of endorigid algebras.

In [BPR16], we prove that almost all the endorigid algebras appearing in Table 2 above are 2-Calabi–Yau tilted. The only exception is the algebra Γ given by the quiver

$$\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & 2 \arrow[loop right, "\gamma"] \end{tikzcd}$$

with relations $\beta\alpha - \gamma\beta, \alpha^2, \gamma^2$. Indeed, this algebra is 1-Gorenstein but we check that it is not stably 3-Calabi–Yau, hence not 2-Calabi–Yau tilted by [KR07, Theorem 3.3]. In fact, this algebra is not stably Calabi–Yau hence not d-Calabi–Yau tilted, for any d.

Definition 1.3.4. A finite dimensional $\mathbb{K}$-algebra is *standard 2-endorigid* if it is isomorphic to the endomorphism algebra of a maximal rigid object in a standard, $\mathbb{K}$-linear, Krull–Schmidt, 2-Calabi–Yau, triangulated category.

Recall that the base field $\mathbb{K}$ is assumed algebraically closed of characteristic zero.

Theorem 1.3.5. *The connected, standard 2-endorigid algebras of finite representation type are exactly the standard 2-Calabi–Yau tilted algebras of finite representation type listed in [BØO11, Theorem 5.7] (see also [Lad14, Section 2.6]) and the non-Jacobian endorigid algebra Γ above.*

It is somewhat disappointing that the word "standard" appears in the statement of Theorem 1.3.5.

Question 1.3.6. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Are all the endorigid algebras of finite representation type over $\mathbb{K}$ standard 2-endorigid?

We give a partial answer to this question: at least, they are all generalised standard.

Definition 1.3.7. A $\mathbb{K}$-linear, Krull–Schmidt, Hom-finite, triangulated category with a Serre functor is called *generalised standard* if all of its morphisms are given by linear combinations of paths in its Auslander–Reiten quiver.

Proposition 1.3.8. *Let $\mathscr{C}$ be a $\mathbb{K}$-linear, Krull–Schmidt, 2-Calabi–Yau, triangulated category. Assume that $T \in \mathscr{C}$ is a cluster tilting object whose endomorphism algebra is generalised standard. Then $\mathscr{C}$ is generalised standard.*

Corollary 1.3.9. *Let $\mathscr{C}$ be a $\mathbb{K}$-linear, Krull–Schmidt, 2-Calabi–Yau, triangulated category. Assume that $T \in \mathscr{C}$ is a cluster tilting object whose endomorphism algebra is of finite representation type. Then $\mathscr{C}$ is generalised standard.*

Chapter 2

Cluster algebras and cluster categories.

In additive categorification of cluster algebras, a Caldero–Chapoton map sends objects in some (cluster) category to Laurent polynomials. It is aimed at inducing a bijection between certain specific objects and cluster variables, cluster monomials, or clusters. Due to the recursive nature of cluster algebras, sending (reachable) indecomposable rigid objects to cluster variables can be nicely encoded in the following three properties that a Caldero–Chapoton map should satisfy. Assume that $\mathscr{C}$ is a skeletally small, $\mathbb{K}$-linear, Hom-finite, Krull–Schmidt, 2-Calabi–Yau triangulated category with a basic cluster tilting object $T = T_1 \oplus \cdots \oplus T_n$, and that CC is a map sending isomorphism classes of objects to Laurent polynomials in $\mathbb{Q}(x_1, \ldots, x_n)$.

(1) For any $i = 1, \ldots, n$,
$$CC(T_i) = x_i$$

(2) For any two objects X, Y,
$$CC(X \oplus Y) = CC(X)CC(Y)$$

(3) For any two objects X, Y such that $\dim_{\mathbb{K}} \operatorname{Ext}^1(X, Y) = 1$,
$$CC(X)CC(Y) = CC(E) + CC(E')$$
where $X \to E \to Y \to \Sigma X$ and $Y \to E' \to X \to \Sigma Y$ are non-split triangles.

Since the original [GLS07, CC06], that respectively have 44 and 162 citations on MathSciNet, constructing Caldero–Chapoton maps in various generality has attracted a lot of research, see for example [CK08, CK06, BMRT07, Pal08, XX10, XX09, Xu10, FK10, Pla11b, Rup11, Dem11, DX12, DG14, ZZ14, HJ15, DSC15, BR15, HJ16, Wil16, GLS18c, Pes18]

In the first two sections of this chapter, we present two such instances of Caldero–Chapoton maps: This first one, introduced in collaboration with Peter Jørgensen, is adapted to cluster-tilting subcategories with possibly infinitely many isoclasses of indecomposables [JP13]. The second one presents the results obtained so far in a work in progress with Pierre-Guy Plamondon: Our main, yet unachieved, aim is to find a direct proof, based on cluster categories, of the multiplication formula for the cluster character of Christof Geiß, Bernard Leclerc and Jan Schröer in [GLS18c].

In a third section, we present some application of cluster categories to the study of the type cone of the $\boldsymbol{g}$-vector fan of cluster algebras of finite type. This is a new approach to a problem that emerged from mathematical physics [AHBHY18] and was brought to light to representation theorists in a talk by Hugh Thomas during the conference *Cluster algebras: twenty years on*, held at the CIRM in 2018 (see the article [BMDM+18], a second version of which, that will also include Nathan Chapelier as a co-author, is expected soon).

The final section of this chapter discusses a work in progress with Peter Jørgensen. Over a cluster-tilted algebra, the support τ-tilting modules of [AIR14] are precisely the images of cluster-tilting objects in the cluster category. However, the definition of a support τ-tilting module makes sense over any finite-dimensional algebra. In that more general setting, they are the images of the 2-term silting complexes in the homotopy category of projectives. In Section 2.4, we adopt a similar point of view with cluster-tilting objects being replaced by cotorsion pairs.

2.1 A Caldero–Chapoton map for infinite clusters.

In this section, we report on the article [JP13] in collaboration with Peter Jørgensen. We check that the Caldero–Chapoton formula also makes sense for cluster tilting subcategories with infinitely many isomorphism classes of indecomposable objects. However, it does not always yield a cluster character, but only a (weak) cluster map in the sense of [BIRS09]: it is generally not defined on every object. We also study the interaction with Iyama–Yoshino reductions, and we apply our results to the cluster category of type A_∞, thus giving examples of infinite friezes.

2.1.1 The Caldero–Chapoton formula for cluster-tilting subcategories.

Let $\mathbb{K}$ be an algebraically closed field, that is assumed of characteristic zero for simplicity. Let $\mathscr{C}$ be a $\mathbb{K}$-linear, Hom-finite, 2-Calabi–Yau, Krull–Schmidt, triangulated category with a cluster tilting subcategory $\mathcal{T}$.

Notation 2.1.1. For any $X \in \mathscr{C}$, we let $GX = \mathscr{C}(-, \Sigma X)|_{\mathcal{T}}$. This yields a functor from $\mathscr{C}$ to the category $\operatorname{mod}\mathcal{T}$ of finitely presented $\mathcal{T}$-modules, *i.e.* of cokernels of natural transformations between representable functors $\mathcal{T}(-, T)$, $T \in \mathcal{T}$. We write $\operatorname{fl}\mathcal{T}$ for the abelian category of modules of finite length over $\mathcal{T}$.

We note that, since $\mathscr{C}$ has weak kernels, the category $\operatorname{mod}\mathcal{T}$ is abelian. Moreover, the functor G induces an equivalence of categories $\mathscr{C}/(\mathcal{T}) \xrightarrow{\simeq} \operatorname{mod}\mathcal{T}$.

Definition 2.1.2. For any $X \in \mathscr{C}$, by [KR07], there are triangles

$$T_1 \to T_0 \to X \to \Sigma T_1 \text{ and } \Sigma T^1 \to X \to \Sigma^2 T^0 \to \Sigma^2 T^1.$$

Let $\operatorname{ind}_{\mathcal{T}} X = [T_0] - [T_1]$ and $\operatorname{coind}_{\mathcal{T}} X = [T^0] - [T^1]$ in the split Grothendieck group $K_0^{\mathrm{sp}}(\mathcal{T})$, be the *index* and *coindex* of X.

Indices were studied in [Pal08, DK08], related to $\boldsymbol{g}$-vectors in [Pla11a]. We note the recent generalisation to higher homological algebra [OT12, Jør18, Rei19].

Lemma 2.1.3. *There is a well-defined group homomorphism*

$$\begin{aligned} K_0(\operatorname{mod} \mathcal{T}) &\xrightarrow{\theta_{\mathcal{T}}} K_0^{\mathrm{sp}}(\mathcal{T}), \\ [GX] &\longmapsto \operatorname{coind}_{\mathcal{T}} \Sigma X - \operatorname{ind}_{\mathcal{T}} \Sigma X. \end{aligned}$$

Definition 2.1.4. Let $X \in \mathscr{C}$ be such that the module GX has finite length. The *Caldero–Chapoton formula* is

$$\phi^{\mathcal{T}}(X) = \underline{x}^{-\operatorname{coind}_{\mathcal{T}} \Sigma X} \sum_{e \in K_0(\operatorname{fl} \mathcal{T})} \chi(\operatorname{Gr}_e(GX)) \underline{x}^{\theta_{\mathcal{T}}(e)} \quad \in \quad \mathbb{Q}(x_T)_{T \in \mathcal{T}},$$

where χ denotes the singular Euler characteristic, and $\operatorname{Gr}_e(GX)$ is the Grassmannian of submodules M of finite length with $[M] = e$.

The Caldero–Chapoton formula satisfies some multiplication formulas that makes it a good candidate for being a (weak) cluster map.

Proposition 2.1.5. *Let $X, Y \in \mathscr{C}$ be such that GX and GY are of finite length. Then:*

1. *We have $\phi^{\mathcal{T}}(X \oplus Y) = \phi^{\mathcal{T}}(X)\phi^{\mathcal{T}}(Y)$*
2. *If $\dim \operatorname{Ext}^1_{\mathscr{C}}(X, Y) = 1$, and $X \to E \to Y \to \Sigma X$, $Y \to E' \to X \to \Sigma Y$ are non-split, then $\phi^{\mathcal{T}}(X)\phi^{\mathcal{T}}(Y) = \phi^{\mathcal{T}}(E) + \phi^{\mathcal{T}}(E')$.*

Theorem 2.1.6. *Assume moreover that the cluster tilting subcategories that can be reached from $\mathcal{T}$ form a cluster structure in the sense of* [BIRS09]*. Let $\mathcal{R}$ be the full additive subcategory of $\mathscr{C}$ whose indecomposable objects are the indecomposables belonging to reachable cluster tilting subcategories. Then:*

- *The Caldero–Chapoton map is a (weak) cluster map $\phi^{\mathcal{T}} : \operatorname{obj} \mathcal{R} \to \mathbb{Q}(x_T)_{T \in \mathcal{T}}$.*
- *In particular, if $\mathcal{T}$ has countably many indecomposable objects up to isomorphism and $\mathcal{A}$ is the cluster algebra associated with the quiver $Q_{\mathcal{T}}$ of $\mathcal{T}$, then $\phi^{\mathcal{T}}$ induces a surjection from $\operatorname{obj} \mathcal{R}$ to the cluster variables of $\mathcal{A}$, sending reachable cluster tilting subcategories to clusters.*

2.1.2 Iyama–Yoshino reductions.

We keep the setup of the previous section.

Definition 2.1.7. Let $\mathcal{U}$ be a functorially finite rigid subcategory of $\mathscr{C}$. The Iyama–Yoshino reduction of $\mathscr{C}$ with respect to $\mathcal{U}$ is the ideal subquotient $\mathscr{C}_{\mathcal{U}} = {}^{\perp}(\Sigma\mathcal{U})/(\mathcal{U})$.

Remark 2.1.8. The typical subcategory $\mathcal{U}$ will be a full additive subcategory of the cluster tilting subcategory $\mathcal{T}$ obtained by removing only finitely many isomorphism classes of indecomposable objects.

By [IY08], $\mathscr{C}_{\mathcal{U}}$ is a $\mathbb{K}$-linear, Hom-finite, 2-Calabi–Yau, Krull–Schmidt, triangulated category. Its shift functor $\Sigma_{\mathcal{U}}$ is given by the choice of triangles $X \to U_X \to \Sigma_{\mathcal{U}} X \to \Sigma X$ with $U_X \in \mathcal{U}$, for each $X \in {}^{\perp}\Sigma\mathcal{U}$.

Proposition 2.1.9. *Let $\mathcal{U}$ be a functorially finite rigid subcategory of $\mathscr{C}$, contained in the cluster tilting subcategory $\mathcal{T}$. Then, there is a commutative diagram*

$$\begin{tikzcd}
\mathscr{C}_{\mathcal{U}}/(\mathcal{T}) \arrow[r, hook] \arrow[d, "\simeq"'] & \mathscr{C}/(\mathcal{T}) \arrow[d, "\simeq"'] \\
\operatorname{mod}\mathcal{T}/(\mathcal{U}) \arrow[r, hook, "\pi^*"'] & \operatorname{mod}\mathcal{T}
\end{tikzcd}$$

where:

1. *The functor π^* is fully faithful and exact.*

2. *The functor π^* induces an equivalence of categories between finitely presented $\mathcal{T}/(\mathcal{U})$-modules and finitely presented $\mathcal{T}$-modules vanishing on $\mathcal{U}$.*

Theorem 2.1.10. *Let $\mathcal{U}$ be a full additive subcategory of $\mathcal{T}$ that is functorially finite. Assume that the cluster tilting subcategories of $\mathscr{C}_{\mathcal{U}}$ that can be reached from $\mathcal{T}$ form a cluster structure in $\mathscr{C}_{\mathcal{U}}$ (this holds for instance when the reachable cluster tilting subcategories form a cluster structure in $\mathscr{C}$). Then the Caldero–Chapoton map $\phi^{\mathcal{T}/(\mathcal{U})}$ for $\mathscr{C}_{\mathcal{U}}$ coincides with the specialization $\phi^{\mathcal{T}}|_{x_U=1 \text{ for each } U\in\mathcal{U}}$.*

Remark 2.1.11. By making use of Proposition 2.1.9 and Theorem 2.1.10, we give a set of sufficient conditions so that the two Caldero–Chapoton formulas actually coincide on the subcategory ${}^{\perp}(\Sigma\mathcal{U}) \cap {}^{\perp}(\Sigma^2\mathcal{U})$, without specializing any variable.

2.1.3 The cluster category of type A_∞.

The cluster category $\mathscr{C}_{\mathrm{A}_\infty}$ of type A_∞ was introduced and studied by Thorsten Holm and Peter Jørgensen in [HJ12]. Its indecomposable objects are in bijection with the arcs, written (m, n) with $m \neq n \in \mathbb{Z}$, of an infinity-gon. It has cluster-tilting subcategories that correspond to triangulations of the infinity-gon that either are *locally finite* (for each n, it

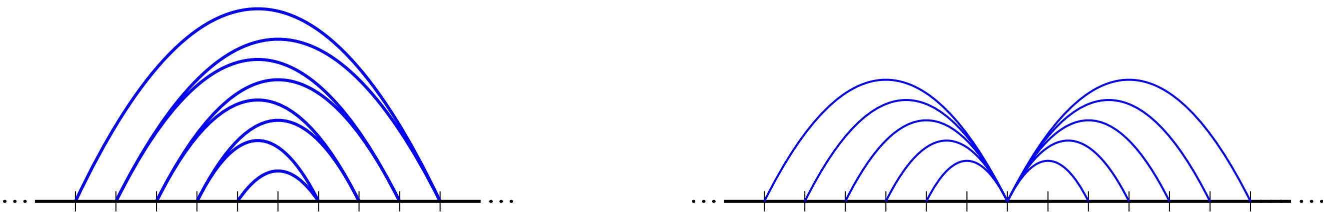


Figure 2.1: An example of a locally finite triangulation of the infinity-gon (left) and of a fountain (right).

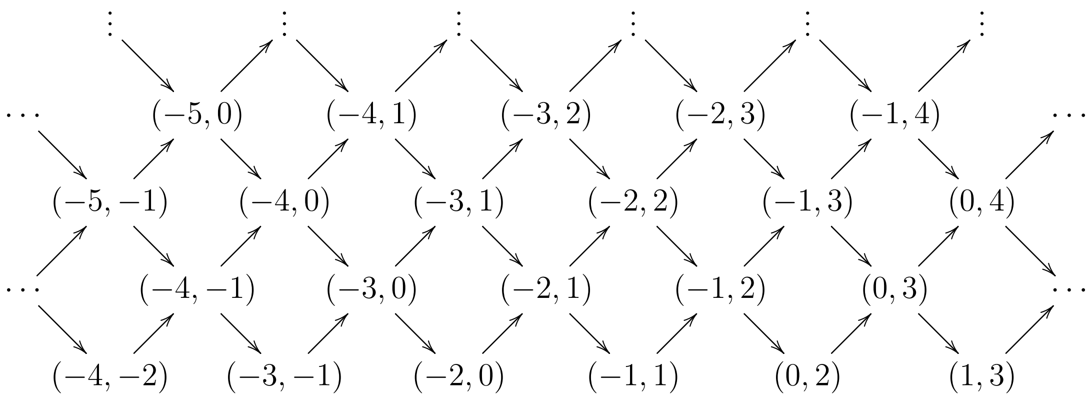


Figure 2.2: A bijection between the arcs of the infinity-gon and the indecomposable objects of $\mathscr{C}_{\mathrm{A}_\infty}$, up to ismomorphism.

contains only finitely many arcs in n) or have a *fountain* (it contains infinitely many arcs of the form (m, n) and infinitely many arcs (n, p), for some n). See Figure 2.1.

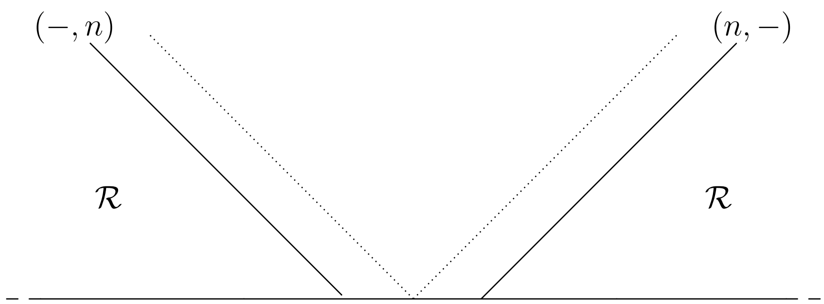
The Auslander–Reiten quiver of the cluster category $\mathscr{C}_{\mathrm{A}_\infty}$ is of shape $\mathbb{Z}\mathrm{A}_\infty$. A bijection between arcs and indecomposables is illustrated in Figure 2.2.

We first describe the domain on which the Caldero–Chapoton map is well-defined.

Theorem 2.1.12. *Let $\mathfrak{T}$ be the triangulation of the infinity-gon associated with the cluster tilting subcategory $\mathcal{T}$ and consider the cluster map $\phi : \operatorname{obj} \mathcal{R} \to \mathbb{Q}(x_T)|_{T \in \operatorname{ind} \mathcal{T}}$. The subcategory $\mathcal{R}$ is determined as follows.*

1. *If $\mathfrak{T}$ is locally finite, then $\mathcal{R} = \mathscr{C}_{\mathrm{A}_\infty}$.*

2. *If $\mathfrak{T}$ has a fountain at n, then $\mathcal{R}$ is* add *of the indecomposable objects which are on or below one of the halflines $(-, n)$ and $(n, -)$ in the AR quiver of $\mathscr{C}_{\mathrm{A}_\infty}$.*



Our main result on the cluster category of type A_∞ is:

Theorem 2.1.13. *The cluster map $\phi^{\mathcal{T}} : \operatorname{obj} \mathcal{R} \to \mathbb{Q}(x_T)_{T \in \operatorname{ind} \mathcal{T}}$ enjoys the following properties.*

1. *$\phi^{\mathcal{T}}(T) = x_T$ for $T \in \operatorname{ind} \mathcal{T}$.*

2. *If $X \in \mathcal{R}$, then $\phi^{\mathcal{T}}(X)$ is a non-zero Laurent polynomial.*

3. *In each such Laurent polynomial, the coefficients in the numerator are positive integers.*

Our strategy for proving Theorem 2.1.13 is to identify some specific Iyama–Yoshino reductions of $\mathscr{C}_{\mathrm{A}_\infty}$ that are equivalent to cluster categories of type A_n so as to make use of Theorem 2.1.10.

Remark 2.1.14. When $\mathcal{T}$ corresponds to the fountain of Figure 2.1, then $\phi^{\mathcal{T}}$ is not a strong cluster map (*i.e.* defined on every object) for a good reason. Indeed, we show that there is no cluster map defined on $\operatorname{obj} \mathscr{C}_{\mathrm{A}_\infty}$ that satisfy the three properties of Theorem 2.1.13.

As a consequence of our results, we can compute some infinite version of friezes, adapted to the Auslander–Reiten quiver of $\mathscr{C}_{\mathrm{A}_\infty}$.

Definition 2.1.15. A *frieze* of the half plane $Q = \{(m,n) \in \mathbb{Z}^2 \mid m \leq n-2\}$ is a map $r : Q \to \mathbb{Z}$ such that:

1. For all $(i,j) \in \mathbb{Z}^2$ with $i+1 \leq j-2$, $r(i,j)r(i+1,j+1) - r(i,j+1)r(i+1,j) = 1$.
2. For all $i \in \mathbb{Z}$, $r(i,i+2)r(i+1,i+3) - r(i,i+3) = 1$.

Theorem 2.1.16. *Let $\mathfrak{T}$ be a locally finite triangulation of the infinity-gon. Then there is a frieze $r : Q \to \mathbb{Z}_{>0}$ such that $r(\mathfrak{t}) = 1$ for each $\mathfrak{t} \in \mathfrak{T}$.*

The frieze of Theorem 2.1.16 is obtained by specializing all variables x_T to 1 in the Caldero–Chapoton map $\phi^{\mathcal{T}}$, with $\mathcal{T}$ the cluster tilting subcategory associated with the triangulation $\mathfrak{T}$.

Remark 2.1.17. A vast "generalisation" of Theorem 2.1.16 and of its converse can be found in [BHJ17].

2.2 A multiplication formula for symmetrizable Cartan matrices.

In a series of articles [GLS17, GLS18a, GLS16, GLS18b, GLS18c, GLS18d], Christof Geiß, Bernard Leclerc and Jan Schröer introduced and studied a family of finite-dimensional algebras, akin to species, but defined by quivers with relations and over any field. They are produced from the data of a symmetrizable Cartan matrix and of the choice of a symmetrizer. Those algebras are of infinite representation type, and their representation theory is seemingly quite badly behaved. However, if one only considers their *locally free* representations, then they behave much like hereditary algebras. In [GLS18c], it is shown that the categories of locally free representations can serve as a categorification for non-necessarily simply-laced cluster algebras of finite type. The authors introduce a Caldero–Chapoton map adapted to that setup. However, their approach to categorification heavily relies on Dynkin types as it makes use of the description [GLS16] of the enveloping algebra $U(\mathfrak{n})$ as a convolution algebra of constructible functions on representation varieties of locally free modules. Our main objective in [PP] is to propose a more direct approach, based on multiplication formulas for the Caldero–Chapoton map, and to construct cluster categories associated with symmetrizable Cartan matrices. We prove a multiplication formula similar to that of Philippe Caldero and Frédéric Chapoton in [CC06] for meshes in the category of locally free representations. Unfortunately, this formula is not sufficient for our categorification purpose: In that context, even for finite types, not all cluster variables can be obtained by considering only almost-split exchange sequences.

2.2.1 Locally-free modules for symmetrizable Cartan matrices.

We first recall the algebras introduced by Christof Geiß, Bernard Leclerc and Jan Schröer in [GLS17, Section 1.4], and some of their results.

Fix a positive integer n.

Definition 2.2.1. A *symmetrizable generalized Cartan matrix* is an $n \times n$ integer matrix $C = (c_{i,j})$ such that

- $c_{i,i} = 2$ for all i;
- $c_{i,j} \leq 0$ if $i \neq j$;
- $c_{i,j} \neq 0$ if, and only if, $c_{j,i} \neq 0$;
- there exists a diagonal matrix $D = \mathrm{diag}(c_1, \ldots, c_n)$ with positive integer coefficients and such that DC is symmetric.

Notation 2.2.2. We let: $\quad g_{i,j} := |\gcd(c_{i,j}, c_{j,i})|, \qquad f_{i,j} := |c_{i,j}|/g_{i,j}.$

Definition 2.2.3. An *orientation* of C is a choice of $\Omega \subset \{1, 2, \ldots, n\}^2$ such that

- $\Omega \cap \{(i,j),(j,i)\} \neq \emptyset$ if, and only if, $c_{i,j} < 0$;
- if the pairs $(i_1, i_2), (i_2, i_3), \ldots, (i_t, i_{t+1})$ all belong to Ω, then $i_1 \neq i_{t+1}$.

Choosing an orientation of C is equivalent to choosing an *acyclic* orientation of the graph with n vertices, where i and j are joined by an edge whenever $c_{i,j} < 0$.

For a skew-symmetrizable generalized Cartan matrix C with an orientation Ω, define the quiver $Q = Q(C, \Omega)$ as follows. The vertices of Q are the integers $1, \ldots, n$. The arrows of Q are of two kinds:

- for all vertices i, there is a loop $\varepsilon_i : i \to i$;
- for all $(i,j) \in \Omega$ and all $1 \leq g \leq g_{i,j}$, an arrow $\alpha_{i,j}^{(g)} : j \to i$.

Finally, if D is a symmetrizer of C, then define the algebra $H = H(C, D, \Omega)$ to be the quotient of the path algebra kQ by the ideal generated by the following relations:

- for all vertices i, $\quad \varepsilon_i^{c_i} = 0$;
- for all $(i,j) \in \Omega$ and each $1 \leq g \leq g_{i,j}$, $\quad \varepsilon_i^{f_{j,i}} \alpha_{i,j}^{(g)} = \alpha_{i,j}^{(g)} \varepsilon_j^{f_{i,j}}$.

Let $H = H(C, D, \Omega)$ be as above. For each vertex i, let $H_i = k[\varepsilon_i] \cong k[X]/(X^{c_i})$.

Definition 2.2.4. A representation M of H is *locally free* if for every vertex i, the H_i-module Me_i is free. Denote by $\mathrm{rep}_{\mathrm{l.f.}}(H)$ the full subcategory of $\mathrm{mod}(H)$ consising of all the locally free modules. In that case, the *rank vector* of M is

$$\underline{\mathrm{rank}}(M) := \left(\mathrm{rank}_{H_i}(Me_i)\right)_{i=1,\ldots,n}.$$

Finally, a locally free module M is *τ-locally free* if for all integers n, $\tau^n M$ is locally free.

The following theorem explains why considering locally free representations should be natural, and it will allow us to associate a cluster category with the algebra $H(C, D, \Omega)$.

Theorem 2.2.5 (Theorem 1.1 of [GLS17]). *The algebra H is 1-Iwanaga–Gorenstein, and the following properties are equivalent for an H-module M:*

1. *M has projective dimension at most 1;*
2. *M has injective dimension at most 1;*
3. *M has finite projective dimension;*
4. *M has finite injective dimension;*
5. *M is locally free.*

We cite one more theorem that illustrates that categories of locally free modules over the algebra H behaves quite similarly to the categories of all representations over hereditary algebras.

Theorem 2.2.6 (Theorem 1.2 of [GLS17]). *Assume that C is connected. Then there are only finitely many isomorphism classes of τ-locally free H-modules if, and only if, the matrix C is of Dynkin type. In that case:*

1. *The map $M \mapsto \underline{\mathrm{rank}}(M)$ gives a bijection between the indecomposable τ-locally free H-modules (up to isomorphism) and the positive roots of the complex semisimple Lie algebra associated to C;*
2. *If M is an indecomposable H-module, then the following are equivalent:*
   (a) M is preprojective;
   (b) M is preinjective;
   (c) M is τ-locally free;
   (d) M is locally free and rigid.

The following fact is quite useful.

Proposition 2.2.7. (Laurent Demonet [GLS18d, Lemma 6.2]) *Any τ-rigid H-module is locally free.*

2.2.2 Cluster categories for symmetrizable Cartan matrices.

In case the symmetrizer is the identity, cluster categories where defined in [BMR+06] as orbit categories of the derived categories of representations of the corresponding quiver. They were shown to be triangulated by Bernhard Keller in [Kel05] who used dg-enhancements in order to construct the “triangulated hull” of an orbit category. Because they also include indecomposable objects corresponding to the cluster variable of the initial seed of a cluster algebra, cluster categories are a natural setup for the Caldero–Chapoton map to live in. In [PP], we apply Bernhard Keller's approach in order to define cluster categories for symmetrizable Cartan matrices.

Because the algebras H of Geiß–Leclerc–Schröer are of infinite global dimension, we cannot make use of the bounded derived category as in [Kel05]. Instead, we are led to consider the perfect derived category (the two categories coincide for finite global dimension). The construction of [Kel05], recalled below, readily applies to perfect derived categories of algebras that are 1-Iwanaga–Gorenstein, a property that is satisfied by the algebras H.

Let $H = H(C, D, \Omega)$ be as in Section 2.2.1, and let $\mathcal{D}(H)$ be the (unbounded) derived category of H and $\mathrm{per}(H)$ be its perfect derived category: the smallest triangulated full subcategory of $\mathcal{D}(H)$ containing H.

By [Hap91], the Nakayama functor $\nu = ? \otimes^L_H DH : \mathcal{D}(H) \to \mathcal{D}(H)$ restricts to an equivalence

$$\nu : \mathrm{per}(H) \xrightarrow{\simeq} \mathrm{per}(H).$$

Define

$$F = \nu \circ [-2] : \mathrm{per}(H) \to \mathrm{per}(H).$$

The functor F enjoys the following properties that are key to constructing the triangulated hull:

Lemma 2.2.8. *Let F be the autoequivalence $\nu \circ [-2]$ of $\mathrm{per}(H)$. Then*

1. *There is a complex of H-H-bimodules X such that $? \otimes^L_H X$ is isomorphic to F, where:*
   - *(a) the complex X is concentrated in degrees 1 and 2;*
   - *(b) the components of X are projective both as left and as right H-modules;*
   - *(c) there exists a quasi-isomorphism $X \xrightarrow{\sim} DH[-2]$.*
2. *For any $C, D \in \mathrm{per}(H)$, the space $\mathcal{D}(H)(C, F^n D)$ vanishes for all but finitely many $n \in \mathbb{Z}$.*

Definition 2.2.9. The *orbit category* $\mathrm{per}(H)/F$ as the same objects as $\mathrm{per}(H)$. For any two objects $C, D \in \mathrm{per}(H)$, we let

$$(\mathrm{per}(H)/F)(C, D) := \bigoplus_{n \in \mathbb{Z}} \mathrm{per}(H)(C, F^n D).$$

The composition of two morphisms $f \in \mathrm{per}(H)(C, F^n D)$ and $g \in \mathrm{per}(H)(D, F^p E)$ is given by $(F^n g) \circ f$.

Even though $\mathrm{per}(H)$ is triangulated, the orbit category generally is not (see [Kel05, Section 3] for counter-examples) which necessitates to embed it into a "smallest" triangulated category: its triangulated hull. Note however that the triangulated hull does not satisfy any universal property at the level of triangulated categories. To obtain a universal property, one needs to consider enhancements and enter the world of dg-categories.

Let $\mathscr{H}$ be the dg-category of bounded complexes of projective H-modules. Then $H^0(\mathscr{H})$ is equivalent to $\mathrm{per}(H)$, and tensoring by X defines a dg-functor $\tilde{F} : \mathscr{H} \to \mathscr{H}$. Note that $\tilde{F}$ is generally not an equivalence of dg-categories, which explains the form of the morphism spaces in the definition below.

Definition 2.2.10. [Kel05] The *dg orbit category* $\tilde{\mathscr{H}}$ is the dg-category whose objects are those of $\mathscr{H}$, and in which the morphism space between two objects A and B is

$$\tilde{\mathscr{H}}(A, B) \cong \mathrm{colim}_p \bigoplus_{n \geq 0} \mathscr{H}(\tilde{F}^n A, \tilde{F}^p B),$$

where the transition maps for the colimit are given by $\tilde{F} : \mathscr{H}(\tilde{F}^n A, \tilde{F}^p B) \to \mathscr{H}(\tilde{F}^{n+1} A, \tilde{F}^{p+1} B)$.

With this definition, there is a canonical functor $\pi : \mathscr{H} \to \tilde{\mathscr{H}}$ which induces an isomorphism $\mathrm{per}(A)/F \cong H^0(\mathscr{H})/F \to H^0(\tilde{\mathscr{H}})$.

Finally, let $\mathrm{per}(\tilde{\mathscr{H}})$ be the smallest full triangulated subcategory of $\mathcal{D}(\tilde{\mathscr{H}})$ containing all representable functors. Then the image of the Yoneda embedding $H^0(\tilde{\mathscr{H}}) \to \mathcal{D}(\tilde{\mathscr{H}})$ is contained in $\mathrm{per}(\tilde{\mathscr{H}})$. Combining this embedding with the above isomorphism, we get an embedding of $\mathrm{per}(A)/F$ into the triangulated category $\mathrm{per}(\tilde{\mathscr{H}})$.

Definition 2.2.11. [Kel05] The category $\mathrm{per}(\tilde{\mathscr{H}})$ is called the *triangulated hull* of the orbit category $\mathrm{per}(A)/F$.

By [Kel05, Section 5.4] the composition

$$\pi : \mathrm{per}(A) \to \mathrm{per}(A)/F \cong H^0(\tilde{\mathscr{H}}) \hookrightarrow \mathrm{per}(\tilde{\mathscr{H}})$$

is an exact functor.

In [PP], we try and justify that $\mathrm{per}(\tilde{\mathscr{H}})$ is a good candidate for a cluster category associated with the algebra H. This is corroborated by some of its properties.

Notation 2.2.12. We write $\mathscr{C}_H$ for $\mathrm{per}(\tilde{\mathscr{H}})$, and call it the *cluster category* of H.

Remark 2.2.13. In case the symmetrizer is the identity, the triangulated hull is the orbit category itself [Kel05, Section 6], and the cluster category $\mathscr{C}_H$ is the usual cluster category of [BMR+06]. At the moment, we do not know whether this remains true for other choices of symmetrizers or not.

Proposition 2.2.14. *The cluster category $\mathscr{C}_H$ is Hom-finite, Krull–Schmidt and the image of H in $\mathscr{C}_H$ generates $\mathscr{C}_H$ as a triangulated category.*

Definition 2.2.15. Let $\mathscr{C}$ be a Hom-finite triangulated category with suspension functor Σ. Then $\mathscr{C}$ is *2-Calabi–Yau* if for any two objects X and Y in $\mathscr{C}$, there is a non-degenerate bilinear form

$$\beta_{X,Y} : \mathscr{C}(X, Y) \times \mathscr{C}(Y, \Sigma^2 X) \longrightarrow k$$

which is functorial in X and in Y, in the sense that if $\phi : X' \to X$ and $\psi : Y \to Y'$ are morphisms, then

$$\beta_{X',Y'}(\psi \circ f \circ \phi, g) = \beta_{X,Y}(f, \Sigma^2 \phi \circ g \circ \psi).$$

Theorem 2.2.16. *The cluster category $\mathscr{C}_H$ is 2-Calabi–Yau.*

We prove this theorem by first lifting the non-degenerate bilinear form on $K^b(\operatorname{proj} H)$ to the dg-category $\mathscr{H}$, then proving that it induces a bilinear form on $\tilde{\mathscr{H}}$.

Unfortunately, the image of H in $\mathscr{C}_H$ does not seem to be cluster-tilting in general. However, the category $\mathscr{C}_H$ satisfies a few more of the properties that hold for cluster categories.

Proposition 2.2.17. *The object πH is rigid in $\mathscr{C}_H$: we have $\mathscr{C}_H(\pi H, \Sigma\pi H) = 0$. Moreover, for any objects $T \in \operatorname{add} H$ and any object $Y \in \mathcal{D}(H)$ which is the cone of a morphism in $\operatorname{add} H$, the functor π induces an isomorphism*

$$\mathcal{D}(H)(T, Y) \xrightarrow{\cong} \mathscr{C}_H(\pi T, \pi Y).$$

Corollary 2.2.18. *The functor $\mathscr{C}_H(\pi H, -)$ induces an equivalence of $\mathbb{K}$-linear categories*

$$\big((\pi H) * \Sigma\pi H\big)/(\Sigma\pi H) \xrightarrow{\simeq} \operatorname{mod} H.$$

Under this equivalence, the suspension functor Σ corresponds to the Auslander–Reiten translation τ.

Remark 2.2.19. All the results stated so far hold more generally when replacing H by any finite-dimensional algebra which is 1-Iwanaga–Gorenstein.

Remark 2.2.20. Specific to the case of the algebra H, there is a well-behaved theory of mutation for maximal rigid objects in $\big((\pi H)*\Sigma\pi H\big)\cap\big((\Sigma^{-1}\pi H)*\pi H\big)$, which lifts τ-tilting mutation from $\operatorname{mod} H$ to $\mathscr{C}_H$.

2.2.3 A multiplication formula.

The definition of cluster characters that we recall here was given in [GLS18c], and generalizes the original definition of [CC06].

Let $H = H(C, D, \Omega)$ be as in Section 2.2.1, and let M be a locally free H-module.

Definition 2.2.21. The *Grassmannian of locally free submodules* of rank vector $\mathbf{r}$ of M, denoted by $\operatorname{Gr}_{\text{l.f.}}(\mathbf{r}, M)$, is the quasi-projective variety whose points parametrize the locally free submodules of M of rank vector $\mathbf{r}$.

Definition 2.2.22 (Definition 1.1 of [GLS18c])**.** The *Caldero–Chapoton map* is the map $X_?$ defined on the set of isomorphism classes of locally free H-modules and with image in $\mathbb{Z}[u_1^{\pm1}, \ldots, u_n^{\pm1}]$ as follows:

$$X_M := \sum_{\mathbf{r}\in\mathbb{N}^n} \chi\big(\operatorname{Gr}_{\text{l.f.}}(\mathbf{r}, M)\big) \prod_{i=1}^{n} v_i^{-\langle \mathbf{r}, \underline{\operatorname{rank}}(E_i)\rangle_H - \langle \underline{\operatorname{rank}}(E_i), \underline{\operatorname{rank}}(M)-\mathbf{r}\rangle_H},$$

where

- $v_i = u_i^{1/c_i}$;
- E_i is a locally free module whose rank vector is the elementary vector ϵ_i (E_i is unique up to isomorphism);
- χ is the Euler–Poincaré characteristic of complex varieties;
- for any locally free modules N and P, $\langle N, P\rangle_H := \dim \operatorname{Hom}_H(N,P) - \dim \operatorname{Ext}_1^1 H(N,P)$. This only depends on the rank vectors of N and P [GLS17, Proposition 4.1], so that $\langle\ ,\ \rangle_H$ is defined on pairs of rank vectors.

Proposition 2.2.23 (Lemma 3.1 of [GLS18c])**.** *Let M and N be two locally free modules. Then $X_{M\oplus N} = X_M \cdot X_N$.*

In the case where the symmetrizer D is the identity matrix, this result was proved in [CC06, Corollay 3.7]. The proof given there does not readily generalize to the case where D is not the identity, and the proof given in [GLS18c] uses a different approach involving an action of $\mathbb{C}^*$ on the Grassmannian. Inspired by these methods, we proved the following generalization of [CC06, Proposition 3.10].

Proposition 2.2.24. *Let M and N be indecomposable locally free H-modules, and let*

$$0 \to M \xrightarrow{i} E \xrightarrow{p} N \to 0$$

be an almost split sequence. Then $X_M \cdot X_N = X_E + 1$.

One of the key steps in the proof of Proposition 2.2.24 is the following equalities relating the Euler characteristics of the Grassmannians of locally free submodules of the terms appearing in an almost-split sequence.

Lemma 2.2.25. *We keep the notations of Proposition 2.2.24. Then for any rank vector $\mathbf{r}$,*

$$\chi\big(\operatorname{Gr}_{\text{l.f.}}(\mathbf{r}, E)\big) = \begin{cases} \sum_{\mathbf{s}+\mathbf{t}=\mathbf{r}} \chi\big(\operatorname{Gr}_{\text{l.f.}}(\mathbf{s}, M)\big)\cdot\chi\big(\operatorname{Gr}_{\text{l.f.}}(\mathbf{t}, N)\big) & \text{if } \mathbf{r} \neq \underline{\operatorname{rank}} N; \\ \sum_{\mathbf{s}+\mathbf{t}=\mathbf{r}} \chi\big(\operatorname{Gr}_{\text{l.f.}}(\mathbf{s}, M)\big)\cdot\chi\big(\operatorname{Gr}_{\text{l.f.}}(\mathbf{t}, N)\big) - 1 & \text{otherwise.} \end{cases}$$

Remark 2.2.26. Our proof also makes use of $\mathbb{C}^*$-actions, and one of the key ideas is to enhance the Grassmannian of locally free submodules N' of N by "fixing a section" $s : N' \to E$ of p.

2.3 The type cone for cluster algebras of finite type.

In this section, reporting on a recent collaboration with Arnau Padrol, Vincent Pilaud and Pierre-Guy Plamondon, we give a new point of view on the articles [AHBHY18, BMDM+18]. Motivated by the geometric approach to scattering amplitudes in mathematical physics, Nima Arkani-Hamed, Yuntao Bai, Song He and Gongwang Yan [AHBHY18] contructed new realizations of the classical associahedra as affine sections of the non-negative orthant $\mathbb{R}^n_{\geq 0}$. Their construction was interpreted in representation-theoretic terms in [BMDM+18] which allowed Véronique Bazier-Matte, Guillaume Douville, Kaveh Mousavand, Hugh Thomas and Emine Yıldırım to give similar (and in fact, all) polytopal realizations of the $\boldsymbol{g}$-vector fans of cluster algebras of simply-laced finite types for acyclic initial seeds. That their constructions give polytopal realizations of $\boldsymbol{g}$-vector fan turns out to follow immediately from two specific properties of the *type cone* of these fans: it has the *unique exchange property* and it is *simplicial*. This is explained in Section 2.3.1, and does not rely on representation theory. We then make use of representation theory and categorification in Section 2.3.2 in order to prove that the type cone of the $\boldsymbol{g}$-vector fans of any cluster algebra of finite type, with respect to any initial seed, satisfies the required properties. A similar approach is used in order to give all polytopal realizations of the $\boldsymbol{g}$-vector fans of (brick and 2-acyclic) gentle algebras, see Section 4.3.2.

2.3.1 The type cone strategy.

The type cone of a polyhedral fan, introduced by Peter McMullen in [McM73], is a polyhedral cone that parametrizes the set of all possible polytopal realizations of the polyhedral fan.

Definition 2.3.1. A *polyhedral cone* is a subset of $\mathbb{R}^n$ that is the positive span of finitely many vectors, or equivalently, that is the intersection of finitely many closed linear half-spaces. A polyhedral cone is *simplicial* when it is given by linearly independent vectors. The intersections of a polyhedral cone with its supporting hyperplanes are called its *faces*. Its *rays* are its 1-dimensional faces and its *facets* are its codimension one faces.

Definition 2.3.2. A *polyhedral fan* is a collection of polyhedral cones that is stable under taking faces and such that the intersection of any two of its cones is a face of both cones. A polyhedral fan is called *simplicial* if all its cones are simplicial, *complete* if the union of its cones is the whole ambient space $\mathbb{R}^n$ and *essential* if it contains $\{0\}$.

Definition 2.3.3. The *normal fan* of a polytope is the collection of the normal cones of all its faces. A polytope P is a *polytopal realization* of $\mathscr{F}$ if $\mathscr{F}$ is the normal fan of P. A fan is called *polytopal* if there exists such a polytopal realization.

Notation 2.3.4. Fix an essential complete simplicial fan $\mathscr{F}$ in $\mathbb{R}^n$. Let $\boldsymbol{G}$ the $N \times n$-matrix whose rows are the rays of $\mathscr{F}$ and let $\boldsymbol{K}$ be a $(N-n) \times N$-matrix that spans the left kernel of $\boldsymbol{G}$ (*i.e.* $\boldsymbol{KG} = 0$). For any height vector $\boldsymbol{h} \in \mathbb{R}^N$, we define the polytope

$$P_{\boldsymbol{h}} := \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{Gx} \leq \boldsymbol{h}\},$$

where $\boldsymbol{u} \leq \boldsymbol{v}$ if and only if for all i, $u_i \leq v_i$.

Definition 2.3.5. The *type cone* of $\mathscr{F}$ is the cone $\mathbb{TC}(\mathscr{F})$ of all *$\mathscr{F}$-admissible* height vectors $\boldsymbol{h} \in \mathbb{R}^N$:

$$\mathbb{TC}(\mathscr{F}) := \left\{\boldsymbol{h} \in \mathbb{R}^N \;\middle|\; \mathscr{F} \text{ is the normal fan of } P_{\boldsymbol{h}}\right\}$$

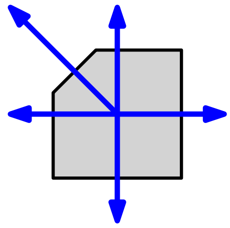


Figure 2.3: The example of a two-dimensional polytope (black) and its normal fan, given by its rays (blue).

Notation 2.3.6. For any two adjacent maximal cones $\mathbb{R}_{\geq 0}\boldsymbol{R}$ and $\mathbb{R}_{\geq 0}\boldsymbol{R}'$ of $\mathscr{F}$ with $\boldsymbol{R} \smallsetminus \{\boldsymbol{r}\} = \boldsymbol{R}' \smallsetminus \{\boldsymbol{r}'\}$, we denote by $\alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})$ the coefficient of $\boldsymbol{s}$ in the unique linear dependence between the rays of $\boldsymbol{R} \cup \boldsymbol{R}'$, *i.e.* such that

$$\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \boldsymbol{R}'} \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})\, \boldsymbol{s} = 0.$$

These coefficients are *a priori* defined up to rescaling, but we additionally fix the rescaling so that $\alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{r}) + \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{r}') = 2$.

Lemma 2.3.7. [CFZ02, Lemma 2.1] *The $\mathscr{F}$-admissible height vectors are characterized by the inequalities:*

$$\mathbb{TC}(\mathscr{F}) = \left\{ \boldsymbol{h} \in \mathbb{R}^N \;\middle|\; \sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \boldsymbol{R}'} \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})\, \boldsymbol{h}_{\boldsymbol{s}} > 0 \;\; \begin{array}{l} \textit{for any adjacent maximal} \\ \textit{cones } \mathbb{R}_{\geq 0}\boldsymbol{R} \textit{ and } \mathbb{R}_{\geq 0}\boldsymbol{R}' \textit{ of } \mathscr{F} \end{array} \right\}$$

Because there are redundancies, not all inequalities above will account for a facet of the type cone, see Figure 2.4.

Definition 2.3.8. A pair of adjacent maximal cones $\{\mathbb{R}_{\geq 0}\boldsymbol{R}, \mathbb{R}_{\geq 0}\boldsymbol{R}'\}$ of $\mathscr{F}$ is called an *extremal adjacent pair* if the associated inequality does account for a facet of the type cone of $\mathscr{F}$.

Remark 2.3.9. Since the type cone has at least $N - n$ facets, it is simplicial when $\mathscr{F}$ has precisely $N - n$ extremal adjacent pairs. This is the property that we aim at proving for cluster algebras and for gentle algebras, by using representation theory, in Sections 2.3.2 and 4.3.2

The polyhedral fans that we will consider will all have another usefull property:

Definition 2.3.10. Two rays $\boldsymbol{r}$ and $\boldsymbol{r}'$ of $\mathscr{F}$ are called *exchangeable* if there are two adjacent maximal cones $\mathbb{R}_{\geq 0}\boldsymbol{R}$ and $\mathbb{R}_{\geq 0}\boldsymbol{R}'$ of $\mathscr{F}$ with $\boldsymbol{R} \smallsetminus \{\boldsymbol{r}\} = \boldsymbol{R}' \smallsetminus \{\boldsymbol{r}'\}$. The polyhedral

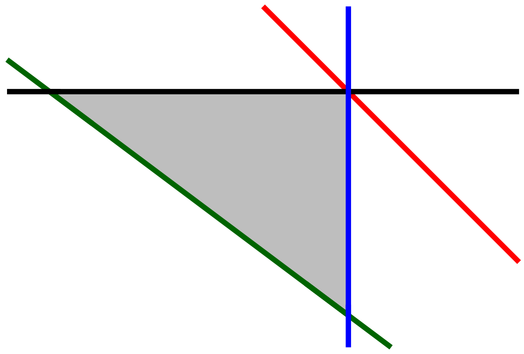


Figure 2.4: The polytope in grey is determined by four inequalities, out of which the blue, green and black ones actually account for facets, while the red one is redundant.

fan $\mathscr{F}$ satisfies the *unique exchange relation property* if, for any exchangeable rays $\boldsymbol{r}, \boldsymbol{r}'$ the linear dependency

$$\sum_{\boldsymbol{s} \in \boldsymbol{R} \cup \boldsymbol{R}'} \alpha_{\boldsymbol{R},\boldsymbol{R}'}(\boldsymbol{s})\, \boldsymbol{s} = 0$$

does not depend on the specific choice of the adjacent maximal cones $\mathbb{R}_{\geq 0}\boldsymbol{R}, \mathbb{R}_{\geq 0}\boldsymbol{R}'$ above but only on $\{\boldsymbol{r}, \boldsymbol{r}'\}$.

Assuming that $\mathscr{F}$ has the unique exchange relation property, its type cone can simply be defined as:

$$\mathbb{TC}(\mathscr{F}) = \left\{ \boldsymbol{h} \in \mathbb{R}^N \;\middle|\; \sum_{\boldsymbol{s}} \alpha_{\boldsymbol{r},\boldsymbol{r}'}(\boldsymbol{s})\, \boldsymbol{h}_{\boldsymbol{s}} > 0 \text{ for any exchangeable rays } \boldsymbol{r} \text{ and } \boldsymbol{r}' \text{ of } \mathscr{F} \right\}.$$

Under the affine map $\mathbb{R}^n \to \mathbb{R}^N$ sending x to $h - \boldsymbol{G}x$, the polytope P_h is sent to the polytope $Q_h = \left\{ z \in \mathbb{R}^N \;\middle|\; \boldsymbol{K}z = \boldsymbol{K}h \text{ and } z \geq 0 \right\}$. This allows for an even simpler reformulation of the type cone, when it is simplicial, which precisely recovers the polytopal realizations of [AHBHY18, BMDM+18]. Notably, all coordinates of $\boldsymbol{z}$ and of $\boldsymbol{\ell}$ below are non-negative.

Corollary 2.3.11. *Assume that the type cone $\mathbb{TC}(\mathscr{F})$ is simplicial and let $\boldsymbol{K}$ be the $(N-n) \times N$-matrix whose rows are the inner normal vectors of the facets of $\mathbb{TC}(\mathscr{F})$. Then the polytope*

$$R_{\boldsymbol{\ell}} := \left\{ \boldsymbol{z} \in \mathbb{R}^N \;\middle|\; K\boldsymbol{z} = \boldsymbol{\ell} \text{ and } \boldsymbol{z} \geq 0 \right\}$$

is a realization of the fan $\mathscr{F}$ for any positive vector $\boldsymbol{\ell} \in \mathbb{R}^{N-n}_{>0}$. Moreover, the polytopes $R_{\boldsymbol{\ell}}$ for $\boldsymbol{\ell} \in \mathbb{R}^{N-n}_{>0}$ describe all polytopal realizations of $\mathscr{F}$.

In the next section, we will prove that when $\mathscr{F}$ is the $\boldsymbol{g}$-vector fan of a cluster algebra of finite type (not necessarily simply-laced) with respect to any seed, then $\mathbb{TC}(\mathscr{F})$ is simplicial. The case of $\boldsymbol{g}$-vector fans of gentle algebras is considered in Section 4.3.2.

2.3.2 Simplicial type cones from cluster categories.

In this section, we specialise $\mathscr{F}$ to be the fan of $\boldsymbol{g}$-vectors for a cluster algebra of (not necessarily simply-laced) finite type, with any given initial seed. For short, this polyhedral fan is called the *cluster fan*. Our aim being to apply Corollary 2.3.11, we remark that the cluster fan $\mathscr{F}$ is already known to have the unique exchange relation property, and we prove that $\mathbb{TC}(\mathscr{F})$ is simplicial.

Let B be any skew-symmetrizable initial exchange matrix, and let $\mathscr{F}_{\mathrm{B}}$ be the associated cluster fan.

This first lemma follows from [BMR+06, Theorem 7.5].

Lemma 2.3.12. *The cluster fan has the unique exchange relation property.*

In order to prove that $\mathbb{TC}(\mathscr{F})$ is simplicial, we describe all extremal exchangeable pairs of $\mathscr{F}$ (see Remark 2.3.9).

Definition 2.3.13. Let $\Sigma = (\mathrm{B}, \mathrm{X})$ be a seed and let $x \in \mathrm{X}$ be a cluster variable. The mutation of Σ at x is a *mesh mutation* that *starts* (resp. *ends*) at x if the entries b_{xy} for $y \in \mathrm{X}$ are all non-negative (resp. all non-positive). A mesh mutation is *initial* if it ends at a cluster variable of an initial seed.

Remark 2.3.14. This definition translates into the language of cluster categories. A mesh mutation starting at X is precisely a mutation such that the exchange triangle $X \to E \to Y \to \Sigma X$ is almost-split. Non-initial mesh mutations thus correspond to meshes of the cluster category that do not end with an indecomposable object in $\operatorname{add} T$, where T is a cluster tilting object corresponding to the initial seed.

Let $\mathcal{M}(\mathrm{B}_\circ)$ denote the set of all pairs $\{x, x'\}$ related by non-initial mesh mutations, and $\mathcal{V}(\mathrm{B}_\circ)$ the set of cluster variables. For $\{x, x'\} \in \mathcal{M}(\mathrm{B}_\circ)$ and $y \in \mathcal{V}(\mathrm{B}_\circ)$, the coefficient $\alpha_{x,x',y}$ is defined as follows: $\alpha_{x,x',y}$ is set to be $|b_{xy}|$ if $y \in \mathrm{X} \cap \mathrm{X}'$ and 0 otherwise, where X, X' are any two clusters such that $\mathrm{X} \smallsetminus \{x\} = \mathrm{X}' \smallsetminus \{x'\}$. By the unique exchange relation property, the coefficients $\alpha_{x,x',y}$ do not depend on the specific choice of clusters X, X'.

Theorem 2.3.15. *Let* B *be a finite type exchange matrix, and let* $\mathscr{F}_{\mathrm{B}}$ *be the associated cluster fan. The extremal exchangeable pairs of the cluster fan* $\mathscr{F}_{\mathrm{B}}$ *correspond to the non-initial mesh mutations.*

This theorem follows from Corollary 2.3.20 below. It implies the following results:

Corollary 2.3.16. *The type cone* $\mathbb{TC}(\mathscr{F}_{\mathrm{B}})$ *is simplicial.*

Corollary 2.3.17. *For any* $\boldsymbol{\ell} \in \mathbb{R}_{>0}^{\mathcal{M}(\mathrm{B}_\circ)}$*, the polytope*

$$R_{\boldsymbol{\ell}}(\mathrm{B}_\circ) := \left\{ \boldsymbol{z} \in \mathbb{R}^{\mathcal{V}(\mathrm{B}_\circ)} \;\middle|\; \boldsymbol{z} \geq 0 \text{ and } \boldsymbol{z}_x + \boldsymbol{z}_{x'} - \sum_{y \in \mathcal{V}(\mathrm{B}_\circ)} \alpha_{x,x',y}\, \boldsymbol{z}_y = \boldsymbol{\ell}_{\{x,x'\}} \text{ for all } \{x, x'\} \in \mathcal{M}(\mathrm{B}_\circ) \right\}$$

is a generalized associahedron, whose normal fan is the cluster fan $\mathscr{F}_{\mathrm{B}}$*. Moreover, the polytopes* $R_{\boldsymbol{\ell}}(\mathrm{B}_\circ)$ *for* $\boldsymbol{\ell} \in \mathbb{R}_{>0}^{\mathcal{M}(\mathrm{B}_\circ)}$ *describe all polytopal realizations of* $\mathscr{F}_{\mathrm{B}}$*.*

In order to prove Theorem 2.3.15, we prove an analogue, for cluster categories, of a result of Maurice Auslander on the Grothendieck group of Artin algebras of finite representation type.

Let $\mathscr{C}$ be a cluster category with finitely many isomorphism classes of indecomposable objects. Let $K_0^{\mathrm{sp}}(\mathscr{C})$ be the split Grothendieck group of $\mathscr{C}$. Fix a cluster-tilting object $T \in \mathscr{C}$, and let $K_0(\mathscr{C}; T)$ be the quotient of $K_0^{\mathrm{sp}}(\mathscr{C})$ by the relations $[X] + [Z] - [Y]$ for all triangles $X \to Y \to Z \xrightarrow{h} \Sigma X$ with $h \in (\Sigma T)$. The reason for considering this specific version of the Grothendieck group is:

Lemma 2.3.18. *Let $X \to Y \to Z \xrightarrow{h} \Sigma X$ be any triangle in $\mathscr{C}$. Then, we have $\operatorname{ind}_T(Y) = \operatorname{ind}_T(X) + \operatorname{ind}_T(Z)$ if and only if h belongs to the ideal (ΣT).*

Denote by $\boldsymbol{g} : K_0^{\mathrm{sp}}(\mathscr{C}) \to K_0(\mathscr{C}; T)$ the canonical projection. For any indecomposable $X \in \mathscr{C}$, write $\ell_X = [X] + [\Sigma^{-1} X] - [E]$, where E is the middle term of an almost split triangle starting at X. For any objects $X, Y \in \mathscr{C}$, we write $\langle X, Y \rangle$ for $\dim_{\mathbb{K}} \operatorname{Hom}_\Lambda(FX, FY)$, where Λ is the cluster-tilted algebra $\operatorname{End}_{\mathscr{C}}(T)$ and F is the equivalence of categories $\mathscr{C}(T, -) : \mathscr{C}/(\Sigma T) \to \operatorname{mod} \Lambda$.

Theorem 2.3.19. *The set $L_{\mathscr{C}} := \{\ell_X \mid X \in \operatorname{ind}(\mathscr{C}) \smallsetminus \operatorname{add}(\Sigma T)\}$ is a basis of the kernel of $\boldsymbol{g}$ and, for any $x \in \ker(\boldsymbol{g})$, we have*

$$x = \sum_{A \in \operatorname{ind}(\mathscr{C}) \smallsetminus \operatorname{add}(\Sigma T)} \frac{\langle x, A \rangle}{\langle \ell_A, A \rangle} \ell_A.$$

Corollary 2.3.20. *Let $X \to E \to Y \xrightarrow{h} \Sigma X$ be a triangle with $h \in (\Sigma T)$. Then the element $x = [X] + [Y] - [E]$ of the kernel of $\boldsymbol{g}$ is a non-negative linear combination of the ℓ_A, with $A \in \operatorname{ind}(\mathscr{C}) \smallsetminus \operatorname{add}(\Sigma T)$.*

2.4 Tau-cotorsion pairs.

2.4.1 Cluster categories and tau-cotorsion pairs.

Let $\mathbb{K}$ be a field and let Λ be a finite-dimensional $\mathbb{K}$-algebra. We write $\operatorname{mod} \Lambda$ for the abelian category of finitely presented Λ-modules, and $\operatorname{proj} \Lambda$ for its full subcategory of projective modules. The functor $\mathbb{D}$ is the $\mathbb{K}$-duality $\mathbb{D}(-) = \operatorname{Hom}_{\mathbb{K}}(-, \mathbb{K})$.

Considering the images in the modules category over a cluster-tilted algebra of all cotorsion pairs in a cluster category leads to the following definition.

Definition 2.4.1. A *τ-cotorsion pair* over Λ is a quadruple $(\mathscr{U}, \mathscr{V}, P, Q)$, where $\mathscr{U}, \mathscr{V}$ are additive subcategories of $\operatorname{mod} \Lambda$ and $P, Q \in \operatorname{proj} \Lambda$ are basic, which satisfies:

(α_0) $\operatorname{add} P = \operatorname{proj} \Lambda \cap {}^{\perp}\mathscr{U}$;

(β_0) $\operatorname{add} Q = \operatorname{proj} \Lambda \cap {}^{\perp}\mathscr{V}$;

(α_1) For any $M \in \operatorname{mod}\Lambda$, we have $M \in \mathscr{U}$ if and only if $\operatorname{Hom}_\Lambda(M, \tau\mathscr{V}) = 0$, $\operatorname{Hom}_\Lambda(\mathscr{V}, \tau M) = 0$ and $\operatorname{Hom}_\Lambda(Q, M) = 0$;

(β_1) For any $N \in \operatorname{mod}\Lambda$, we have $N \in \mathscr{V}$ if and only if $\operatorname{Hom}_\Lambda(N, \tau\mathscr{U}) = 0$, $\operatorname{Hom}_\Lambda(\mathscr{U}, \tau N) = 0$ and $\operatorname{Hom}_\Lambda(P, N) = 0$;

(γ_0) $\mathscr{U}$ is a precovering subcategory of $\operatorname{mod}\Lambda$;

(γ_1) $\mathbb{D}\Lambda$ has an $\operatorname{add}(\tau\mathscr{U} \oplus \nu P)$-precover;

(δ_0) $\mathscr{V}$ is a preenveloping subcategory of $\operatorname{mod}\Lambda$;

(δ_1) Λ has a $\operatorname{add}(\tau^{-1}\mathscr{V} \oplus \nu^{-1}\Sigma^2 Q)$-preenvelope.

Remark 2.4.2. In conditions (α_1) and (β_1), requiring the vanishing of τ-orthogonals in both direction might seem like the wrong thing to do. However, it is reminiscent of Aslak B. Buan and Robert J. Marsh's definition of τ-exceptional sequences [BM18]. Moreover, the symmetry of first extensions in a 2-Calabi–Yau category forces this vanishing in both directions upon us.

Notation 2.4.3. For any additive subcategory $\mathscr{U}$ of $\operatorname{mod}\Lambda$, write $\mathscr{U}^\perp$ for the Hom-perpendicular subcategory and $\mathscr{U}^{\perp_\tau}$ for the full subcategory of all those modules M that satisfy $\operatorname{Hom}_\Lambda(\mathscr{U}, \tau M) = 0$. One defines ${}^{\perp_\tau}\mathscr{U}$ similarly. For any additive subcategory $\mathscr{A} = \mathscr{M} \times \mathscr{P}$ of $\operatorname{mod}\Lambda \times \operatorname{proj}\Lambda$, let $\mathscr{A}^{\perp_{\mathbb{E}_\Lambda}}$ denote the full subcategory of all those pairs $(N, Q) \in \operatorname{mod}\Lambda \times \operatorname{proj}\Lambda$ that satisfy $N \in \mathscr{M}^{\perp_\tau} \cap {}^{\perp_\tau}\mathscr{M} \cap \mathscr{P}^\perp$ and $Q \in \operatorname{proj}\Lambda \cap {}^\perp\mathscr{M}$.

Remark 2.4.4. The notation above allow for an alternative, more concise, definition. A τ-cotorsion pair over Λ is equivalently a pair $(\mathscr{S}, \mathscr{T}) = (\mathscr{U} \times \operatorname{add} P, \mathscr{V} \times \operatorname{add} Q)$ of additive subcategories of $\operatorname{mod}\Lambda \times \operatorname{proj}\Lambda$ satisfying $\mathscr{S} = \mathscr{T}^{\perp_{\mathbb{E}_\Lambda}}$, $\mathscr{T} = \mathscr{S}^{\perp_{\mathbb{E}_\Lambda}}$, ($\gamma_0$), ($\gamma_1$), ($\delta_0$), ($\delta_1$).

The following method supplies a wealth of examples of τ-cotorsion pairs when Λ is representation-finite: Let $\mathscr{A}_0 = \mathscr{M}_0 \times \mathscr{P}_0$ be an additive subcategory of $\operatorname{mod}\Lambda \times \operatorname{proj}\Lambda$. Define $\mathscr{B} = (\mathscr{A}_0)^{\perp_{\mathbb{E}_\Lambda}}$ and $\mathscr{A} = \mathscr{B}^{\perp_{\mathbb{E}_\Lambda}}$. Then the pair $(\mathscr{A}, \mathscr{B})$ satisfies conditions (α_0), (α_1), (β_0), (β_1). When Λ is assumed representation-finite, the remaining conditions appearing in the definition of a τ-cotorsion pair are automatically satisfied. In [JP19], we apply this method in order to classify all τ-cotorsion pairs over some Nakayama algebras, by means of arc diagrams [Ada16, Section 2.2].

We now investigate another method for constructing τ-cotorsion pairs: Those are bijectively related to (usual) cotorsion pairs in 2-Calabi–Yau triangulated categories.

Let $\mathscr{C}$ be a $\mathbb{K}$-linear, Hom-finite, Krull–Schmidt, triangulated 2-Calabi–Yau categories with a cluster tilting object T. Write Λ for the endomorphism algebra of T in $\mathscr{C}$.

Notation 2.4.5. The functor $\mathscr{C}(T, -) : \mathscr{C} \to \operatorname{mod}\Lambda$ induces an equivalence of categories $\mathscr{C}/(\Sigma T) \simeq \operatorname{mod}\Lambda$.

- For an additive subcategory $\mathscr{X}$ of $\mathscr{C}$, we let $\overline{\mathscr{X}}$ denote the image of $\mathscr{X}$ under the functor $\mathscr{C}(T, -)$ and $P^-(\mathscr{X})$ a basic projective Λ-module such that $\operatorname{add} P^-(\mathscr{X}) = \overline{\Sigma^{-1}\mathscr{X} \cap \operatorname{add} T}$.

- Let $(\mathscr{A}, \mathscr{B})$ be a pair of additive subcategories of $\mathscr{C}$. We associate to this pair the quadruple $(\overline{\mathscr{A}}, \overline{\mathscr{B}}, P^-(\mathscr{A}), P^-(\mathscr{B}))$.

Definition 2.4.6. If $\mathscr{U}$ is an additive subcategory of $\operatorname{mod}\Lambda$, we call *special lift* of $\mathscr{U}$ the additive subcategory $\mathscr{A}$ of $\mathscr{C}$ such that $\overline{\mathscr{A}} = \mathscr{U}$ and $\mathscr{A} \cap \operatorname{add}\Sigma T = 0$.

Theorem 2.4.7. *Let $\mathscr{C}$ be a $\mathbb{K}$-linear, Hom-finite, Krull–Schmidt, triangulated 2-Calabi–Yau categories with a cluster tilting object T, and let Λ be the endomorphism algebra of T in $\mathscr{C}$.*

(i) Let $(\mathscr{A}, \mathscr{B})$ be a pair of additive subcategories of $\mathscr{C}$. Then $(\mathscr{A}, \mathscr{B})$ is a cotorsion pair if and only if $(\overline{\mathscr{A}}, \overline{\mathscr{B}}, P^-(\mathscr{A}), P^-(\mathscr{B}))$ satisfies (α_0), (α_1), (β_0), (β_1), (γ_0), (γ_1).

(ii) The correspondence sending the pair $(\mathscr{A}, \mathscr{B})$ of additive subcategories of $\mathscr{C}$ to the quadruple $(\overline{\mathscr{A}}, \overline{\mathscr{B}}, P^-(\mathscr{A}), P^-(\mathscr{B}))$ induces a bijection between cotorsion pairs in $\mathscr{C}$ and τ-cotorsion pairs in $\operatorname{mod}\Lambda$. The inverse bijection is obtained by sending $(\mathscr{U}, P)$ to the additive subcategory of $\mathscr{C}$ generated by $\widetilde{\mathscr{U}}$ and $\Sigma^{-1}P$, where $\widetilde{\mathscr{U}}$ is a special lift of $\mathscr{U}$.

We note the following step in the proof of Theorem 2.4.7, that might be of independent interest.

Proposition 2.4.8. *An additive subcategory $\mathscr{A}$ is precovering in $\mathscr{C}$ if and only if the pair $(\overline{\mathscr{A}}, P^-(\mathscr{A}))$ satisfies conditions $(\gamma_0)$and $(\gamma_1)$of Definition 2.4.1.*

We obtain, as a specific case of Theorem 2.4.7:

Corollary 2.4.9. *The correspondence sending $(\mathscr{A}, \mathscr{B})$ to $(\overline{\mathscr{A}}, \overline{\mathscr{B}})$ induces a bijection between cotorsion pairs $(\mathscr{A}, \mathscr{B})$ such that $\mathscr{A} \cap \operatorname{add}\Sigma^{-1}T = 0$ and $\mathscr{B} \cap \operatorname{add}\Sigma^{-1}T = 0$ and pairs $(\mathscr{U}, \mathscr{V})$ of additive subcategories of $\operatorname{mod}\Lambda$ satisfying:*

(α_0') $\operatorname{proj}\Lambda \cap {}^{\perp}\mathscr{U} = 0$;

(β_0') $\operatorname{proj}\Lambda \cap {}^{\perp}\mathscr{V} = 0$;

(α_1') *For any $M \in \operatorname{mod}\Lambda$, we have $M \in \mathscr{U}$ if and only if $\operatorname{Hom}_\Lambda(M, \tau\mathscr{V}) = 0$ and $\operatorname{Hom}_\Lambda(\mathscr{V}, \tau M) = 0$;*

(β_1') *For any $N \in \operatorname{mod}\Lambda$, we have $N \in \mathscr{V}$ if and only if $\operatorname{Hom}_\Lambda(N, \tau\mathscr{U}) = 0$ and $\operatorname{Hom}_\Lambda(\mathscr{U}, \tau N) = 0$;*

(γ_0) *$\mathscr{U}$ is a precovering subcategory of $\operatorname{mod}\Lambda$;*

(γ_1) *$\mathbb{D}\Lambda$ has a $\tau\mathscr{U}$-precover.*

2.4.2 Tau-cotorsion pairs and two-term silting complexes.

Let $K^{[-1,0]}(\operatorname{proj}\Lambda)$ be the full subcategory $\Lambda * \Sigma\Lambda$ of $\mathrm{K}^b(\operatorname{proj}\Lambda)$. Its objects are the so-called *two-term complexes*, i.e. the complexes concentrated in cohomological degrees -1 and 0. Inspired by the bijection between support τ-tilting modules and two-term silting complexes of projectives [AIR14], we relate τ-cotorsion pairs to Ext-orthogonal pairs of subcategories of $K^{[-1,0]}(\operatorname{proj}\Lambda)$.

We thus have a functor $\mathrm{H}_0(-) : K^{[-1,0]}(\operatorname{proj}\Lambda) \to \operatorname{mod}\Lambda$, inducing an equivalence of categories

$$K^{[-1,0]}(\operatorname{proj}\Lambda)/(\Sigma\Lambda) \simeq \operatorname{mod}\Lambda.$$

With an additive subcategory $\mathscr{X}$ of $K^{[-1,0]}(\operatorname{proj}\Lambda)$, we associate the pair

$$(\mathrm{H}_0(\mathscr{X}), \mathrm{H}_1(\mathscr{X} \cap \operatorname{add}\Sigma\Lambda)).$$

The reason why conditions (γ_0) and (γ_1) are part of the definition of a τ-cotorsion pair is the following proposition that is of independent interest.

Proposition 2.4.10. *Let $\mathscr{X}$ be an additive subcategory of $K^{[-1,0]}(\operatorname{proj}\Lambda)$, and let $(\mathscr{U}, P) = (\mathrm{H}_0(\mathscr{X}), \mathrm{H}_1(\mathscr{X} \cap \operatorname{add}\Sigma\Lambda))$ be the associated pair. Then $\mathscr{X}$ is precovering in $K^{[-1,0]}(\operatorname{proj}\Lambda)$ if and only if $\mathscr{U}$ is precovering in $\operatorname{mod}\Lambda$ and $\mathbb{D}\Lambda$ has an $\operatorname{add}(\tau\mathscr{U} \oplus \nu P)$-precover.*

Theorem 2.4.11. *The correspondence sending an additive subcategory $\mathscr{X}$ of $K^{[-1,0]}(\operatorname{proj}\Lambda)$ to the pair $(\mathrm{H}_0(\mathscr{X}), \mathrm{H}_1(\mathscr{X} \cap \operatorname{add}\Sigma\Lambda))$ induces a bijection between:*

1. *Pairs $(\mathscr{X}, \mathscr{Y})$ of additive subcategories of $K^{[-1,0]}(\operatorname{proj}\Lambda)$ such that*
   - *(a) $\mathscr{X} = {}^{\perp_1}\mathscr{Y} \cap \mathscr{Y}^{\perp_1}$;*
   - *(b) $\mathscr{Y} = {}^{\perp_1}\mathscr{X} \cap \mathscr{X}^{\perp_1}$;*
   - *(c) $\mathscr{X}$ is precovering and*
2. *quadruples $(\mathscr{U}, \mathscr{V}, P, Q)$ where $\mathscr{U}, \mathscr{V}$ are additive subcategories of $\operatorname{mod}\Lambda$, P, Q are in $\operatorname{proj}\Lambda$, satisfying (α_0), (α_1), (β_0), (β_1), (γ_0), (γ_1).*

Moreover $\mathscr{Y}$ is preenveloping if and only if the associated $(\mathscr{U}, \mathscr{V}, P, Q)$ is a τ-cotorsion pair.

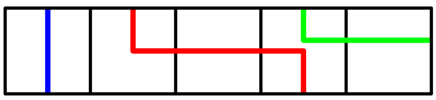
Chapter 3

Tau-tilting theory of gentle algebras.

In this chapter, we categorify two combinatorial notions of flips by using the τ-tilting theory [AIR14] of gentle algebras. We show that the flips of non-kissing walks on a grid [McC17] and the flips of non-crossing accordions [GM18, MP19] on surfaces are both combinatorial shadows of the mutation of support τ-tilting modules over gentle algebras.

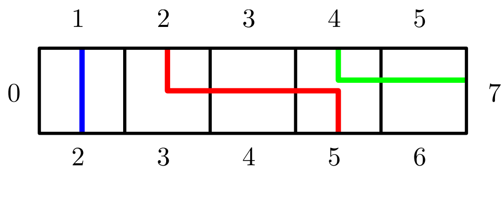
3.1 The non-kissing complex of a gentle algebra.

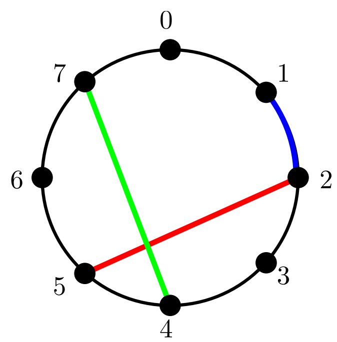
In order to motivate our results in [PPP17], let us first consider a grid in a row:



In that grid, we are allowed to draw maximal paths, called *walks*, going from NW (top and left) towards SE (bottom and right). The path in blue is an example of a *straight* walk, while the two paths in red and green are instances of *bending* walks.

This combinatorics is motivated by the fact that it relates to triangulations of the polygon: Bending walks are in bijection with diagonals of the polygon.





The grid has the advantage over the polygon that it naturally gives an orientation to flips. This grid in a row correspond to an initial triangulation of the form $[0\,2], [0\,3], \ldots,$

while looking at "staircase ribbon grids" correspond to taking different, acyclic, initial triangulations, hence changing the orientation of flips.

Using the bijection between bending walks and diagonals of the polygons allows to transfer the rich combinatorics of triangulations to combinatorics of walks: Crossing, triangulations and flips have natural analogues in the world of walks in the grid.

We note that this combinatorics has been categorified by Philippe Caldero, Frédéric Chapoton and Ralf Schiffler in [CCS06] with cluster categories [BMR+06] of Dynkin type A.

The combinatorics of triangulations of a polygon generalises to (tagged) triangulations of marked Riemann surfaces with boundaries, marked points (and punctures). This was succesfully used in order to study a large class of cluster algebras [FST08]. Some generalised cluster categories [Ami09, BZ11, QZ17] categorify this class of cluster algebras.

In a quite different direction, Thomas McConville generalised the combinatorics of walks to the case of grids of any shape. However, a link to representation theory was missing.

The situation is summarized in the following table:

Combinatorics	**Geometry**	**Representation theory**
Grid in a row	Polygon	Cluster categories of type A
Bending walks	diagonals	indecomposable (rigid) objects up to isomorphism
"kissing"	crossing	non-split extensions
non-kissing facets	triangulations	cluster tilting objects
"flips"	flips	mutation
	Generalisations	
	Marked surfaces	Cluster categories from surfaces
Grids of any shape		?

This justifies our

Aim 3.1.1. Give an algebraic interpretation of non-kissing facets for grids of any shapes. Cross this bridge between combinatorics and representation theory in both directions.

3.1.1 From grids to grid algebras.

In this section, we describe the result by Thomas McConville, alluded to above, that motivated the work in [PPP17]. We also explain some of the results in [PPP17] for the specific case of a grid.

Definition 3.1.2. A *grid* is a non-empty subset of $\mathbb{Z}^2$, whose points will be represented by area 1 squares, centered in that point. Given a fixed grid G, a *walk* in G is a maximal NW to SE path. A *straight walk* is a walk that only goes S or only goes E. A *bending walk* is a non-straight walk, i.e. a walk that bends at least once. There are two families of walks that bend exactly once: the *initial walks*, also called *projective walks* for reasons that will become clear later, are those that bend precisely once, from N to E. Similarly, the *terminal walks*, or shifted projective walks, are those that bend once, from W to S. A

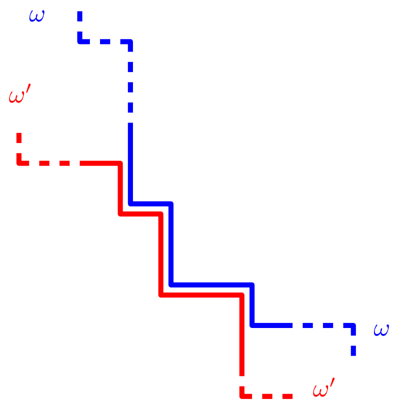


Figure 3.1: The local configuration of a kiss.

walk ω *kisses* a walk ω' if the local configuration of figure 3.1 arises: The two walks share a common subwalk that ω enters from N and leaves towards E, and that ω' enters from W and leaves towards S. Note that kisses are oriented. We will nonetheless say that ω and ω' *kiss* if ω kisses ω' or ω' kisses ω. A *non-kissing facet*, or simply a facet, is a maximal set of pairwise non-kissing walks.

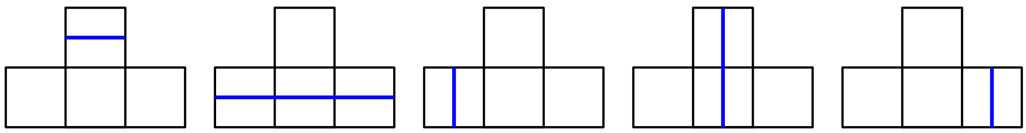
We note a few straightforward facts about kissings.

Remark 3.1.3. 1. Because straight walks do not bend, they do not kiss any walk, nor are they kissed by any walk. We thus often consider *reduced facets*, where all straight walks have been removed. The adjective "reduced" will often be omitted.

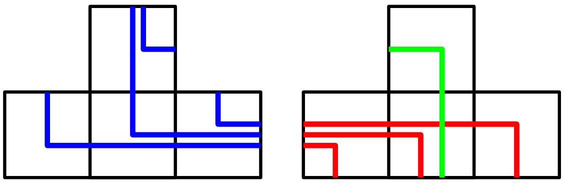
2. Because a kiss involves a walk bending from N to E and a walk bending from W to S, the projective walks are pairwise non-kissing. Similarly, shifted projective walks are pairwise non-kissing.

3. The set of projective walks forms a (reduced) non-kissing facet. This can be seen by remarking that there is a projective walk attached to each square of the grid, leaving no room for a W to S bend not involving a kiss. Similarly, the shifted projective walks form a (reduced) non-kissing facet.

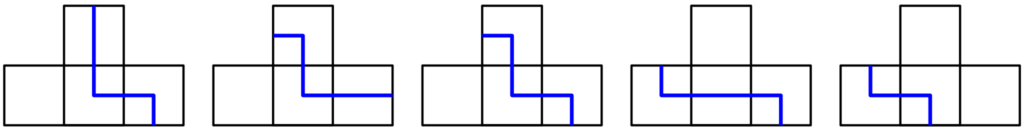
We explicit all walks in a small example. Here is the list of all straight walks in some "tetrix-shaped" grid:



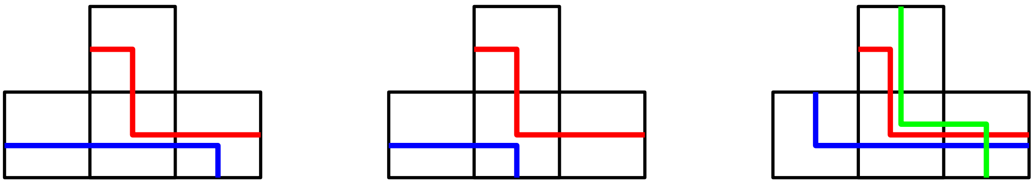
The walks bending exactly once are as follows (projective walks are in blue, while shifted projective walks are in red and green):



The remaining walks bend at least twice and are the following ones:



The next figure illustrates examples of a kiss, of a “shy” kiss, and of three walks that are pairwise non-kissing:



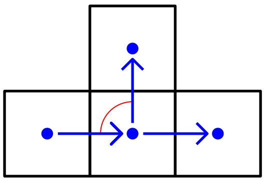
Theorem 3.1.4 (McConville)**.** *Let G be a grid. Then:*

1. *All non-kissing facets have the same cardinality: The number of bending walks in each facet is the number of squares of the grid.*

2. *Let F be a non-kissing facet and let $\omega \in F$ be a bending walk. Then there is a unique other bending walk ω' such that $F' = F \triangle \{\omega, \omega'\}$ is a non-kissing facet.*

Definition 3.1.5. The facet F' of the previous theorem is called the *flip* of F at ω.

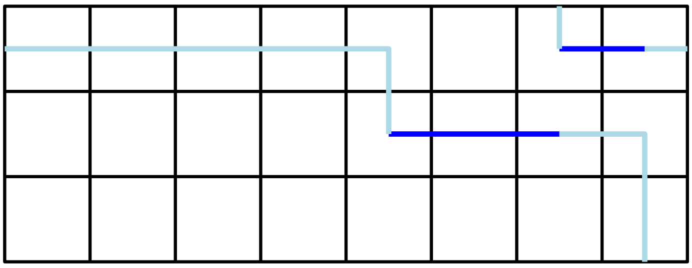
We would like to interpret walks in the grid G as indecomposable representations of some finite-dimensional algebra. Following the type A case, corresponding to a row-grid and to staircase ribbon grids, a first guess is to consider a graph dual to the grid, oriented W to E and S to N. However, representations of that quiver do not reflect the fact that walks can only go from NW to SE. We are thus lead to introduce relations on that quiver, morally preventing representations to go N or W.



Definition 3.1.6. Let G be a grid. The associated *grid algebra* is the basic algebra given by the quiver with relations defined as follows: The quiver of the grid G has set of vertices the squares of G. Two vertices v, w are linked by an arrow if and only if the two squares

are adjacent. In that case, the arrow is oriented from W to E and from S to N. The ideal of relations is generated by quadratic monomials, one for each pair of composable arrows associated with three unaligned squares.

One can associate a walk on G to each indecomposable representation of the associated grid algebra: First draw the quiver on the grid; then draw part of a walk by following the support of the representation on the grid (dark blue) ; finally complete it to a walk by adding one step N then all W (if possible) on one side, and one step E then all S (if possible) on the other side (light blue):



This defines a map $\omega_? : M \mapsto \omega_M$ from the set of indecomposable representations of the grid algebra to the set of walks in G.

Remark 3.1.7. It is immediate from the construction above that the straight walks are not in the image of $\omega_?$. By considering the reduced facets only, this causes no trouble. However, the shifted projective walks are not in the image of $\omega_?$ either. We thus also consider symbols $P_v[1]$, for each vertex v of the grid quiver, that we think of as shifted indecomposable projectives. We extend the map $\omega_?$ by letting $\omega_{P_v[1]}$ be the shifted projective walk that bends in the square v.

Notation 3.1.8. For a grid G, with associated grid algebra Λ, we let:

- $\mathcal{W}_{\text{bend}}(G)$ be the set of all bending walks in G;
- $\operatorname{ind}\Lambda$ be a set of representatives for the isoclasses of indecomposable Λ-modules;
- $\Phi_{\geq -1}(\Lambda) = \operatorname{ind}\Lambda \sqcup \{P_v[1], v \in Q_0\}$ be the set of *almost positive* Λ-modules.

Theorem 3.1.9. *Let G be any grid and let $\Lambda = \mathbb{K}Q/I$ be the associated grid algebra. Then:*

1. *The map $\omega_?$ induces a bijection $\Phi_{\geq -1}(\Lambda) \to \mathcal{W}_{\text{bend}}(G)$.*
2. *For any $M, N \in \Phi_{\geq -1}(\Lambda)$, the walk ω_M kisses ω_N if and only if $\operatorname{Hom}_\Lambda(N, \tau M) \neq 0$.*

Corollary 3.1.10. *The map $\omega_?$ induces a bijection between support τ-tilting modules over Λ and (reduced) non-kissing facets. Under this bijection, flips correspond to support τ-tilting mutation.*

Question 3.1.11. Grid algebras form a subclass of the class of gentle algebras. Is it possible to extend the combinatorics of walks, and Theorem 3.1.9, Corollary 3.1.10 to the class of all gentle algebras? This is our aim in the next section.

3.1.2 Applications to combinatorics via gentle algebras.

Gentle algebras form a class of algebras whose representation theory is combinatorially well-behaved. Their indecomposable representations can be described in terms of strings and bands [BR87], which are certain words in the arrows of a quiver and their formal inverses. Notably, this class of algebras is shown to be stable under derived equivalence by Jan Schröer and Alexander Zimmermann in [SZ03].

In this section, we explain how to generalise the combinatorics of walks from the case of a grid to the case of any gentle algebra.

Definition 3.1.12. A gentle algebra is the algebra $\mathbb{K}Q/I$ of a bound quiver (Q, I) satisfying the following conditions:

- the ideal I is generated by quadratic monomials and is admissible.
- For any arrow $\alpha \in Q_1$, there is at most one arrow β such that $\alpha\beta \in I$ and at most one arrow γ such that $\alpha\gamma \notin I$.
- For any arrow $\alpha \in Q_1$, there is at most one arrow β such that $\beta\alpha \in I$ and at most one arrow γ such that $\gamma\alpha \notin I$.

The main difficulty in tying to generalise the combinatorics of grids to any gentle algebra is to understand what should play the role of the grid for a gentle algebra that does not come from a grid. This is precisely what the blossoming bound quiver is made for. We note that this quiver was independently introduced in [BDM+17] under the name "fringed quiver". The same construction had appeared, prior to both [BDM+17, PPP17], in a survey article by Hideto Asashiba [Asa12]. There, the blossoming bound quiver is used in order to simplify the definition of the Avella-Alaminos–Geiß invariant [AAG08].

Definition 3.1.13. Let (Q, I) be a gentle bound quiver. The *blossoming* bound quiver of (Q, I) is the unique (up to isomorphism) gentle bound quiver $(Q^{❀}, I^{❀})$ each of whose vertex is either a leaf (i.e. of valency one) or four-valent and from which (Q, I) is obtained by removing all leaves. In other words, $(Q^{❀}, I^{❀})$ is obtained from (Q, I) by minimally making all vertices of Q four-valent. A *blossoming vertex*, or simply a *blossom*, is a leaf of $(Q^{❀}, I^{❀})$.

Remark 3.1.14. If the gentle bound quiver (Q, I) is the bound quiver of a grid G, then its blossoming bound quiver is obtained by considering the boundaries of the grid, as in Figure 3.2.

Definition 3.1.15. Let (Q, I) be a gentle bound quiver.

1. A *string* for (Q, I) is a "composable" word in the alphabet given by the arrows of Q and their formal inverses which does not contain any consecutive
    - $\alpha\,\alpha^{-1}$ or $\alpha^{-1}\,\alpha$;

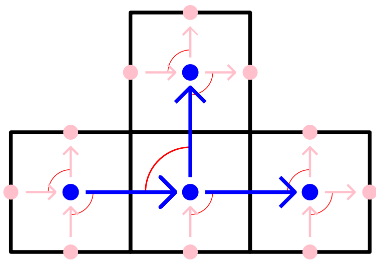


Figure 3.2: The blossoming bound quiver of a grid bound quiver

- $\alpha\,\beta$ or $\beta^{-1}\,\alpha^{-1}$, with $\alpha\,\beta \in I$.

For each vertex $v \in Q_0$, there is also a *string of length zero*, denoted by ε_v, that starts and ends at v. We denote by $\mathcal{S}(Q,I)$ the set of strings on (Q,I). As is usual, we often implicitly identify the two inverse strings ρ and ρ^{-1}.

2. We call *negative simple string* a formal word of length zero of the form $-v$, where v is any vertex of Q_0. We let $\mathcal{S}(Q,I)_{\geq -1} := \mathcal{S}(Q,I) \sqcup \{-v \mid v \in Q_0\}$ be the set of *almost positive strings*. Note that ε_v and $-v$ are different almost positive strings.

3. A *walk* on (Q,I) is a string for $(Q^{❀}, I^{❀})$ that starts and ends in a blossoming vertex. A *straight walk* on (Q,I) is a walk that is a path in $(Q^{❀}, I^{❀})$. A *bending walk* is a walk which is not straight.

Remark 3.1.16. Assuming that (Q,I) comes from a grid, and drawing its blossoming $(Q^{❀}, I^{❀})$ as in Figure 3.2, shows that Definition 3.1.15 generalises the corresponding notions from Section 3.1.1.

Taking profit of the previous remark leads to the following definition:

Definition 3.1.17. Let (Q,I) be a gentle bound quiver and let $(Q^{❀}, I^{❀})$ be its blossoming bound quiver.

1. If ω is a walk on (Q,I), then a *bottom substring* of ω is a string σ for (Q,I) such that $\omega = \rho\alpha\sigma\beta^{-1}\rho'$, for some arrows α, β. Similarly, a *top substring* of ω is a string σ for (Q,I) such that $\omega = \rho\alpha^{-1}\sigma\beta\rho'$, for some arrows α, β. The terminology is motivated by the usual schematic representation of a string, see Figure 3.3.

2. A walk ω *kisses* a walk ω' if there is a string σ for (Q,I) which is at the same time a top substring of ω and a bottom substring of ω'.

3. A *non-kissing facet* of (Q,I) is a maximal set of non self-kissing and pairwise non-kissing walks on (Q,I).

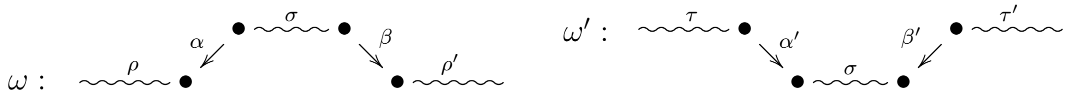


Figure 3.3: Schematic representations of a top substring (left) of a walk ω and of a bottom substring (right) of a walk ω'. Here, the walk ω kisses ω' along σ.

Definition 3.1.18. The *non-kissing complex* $\mathcal{NK}(Q, I)$ is the simplicial complex whose faces are the sets of pairwise non-kissing walks on (Q, I). We note that, by definition, self-kissing walks never appear in $\mathcal{NK}(Q, I)$ and that the straight walks appear in all facets of $\mathcal{NK}(Q, I)$.

In [PPP17], we give two proofs of the following result. One is purely combinatorial, and heavily inspired from the proof of T. McConville, and the other is purely representation theoretic and makes use of τ-tilting theory (via the results of Section 3.1.3).

Theorem 3.1.19. *Let (Q, I) be any gentle bound quiver with blossoming $(Q^{\circledast}, I^{\circledast})$.*

1. *All non-kissing facets of (Q, I) have the same cardinality: The number of bending walks in each facet is the number of vertices of Q.*
2. *Let F be a non-kissing facet of (Q, I) and let $\omega \in F$ be a bending walk. Then there is a unique other bending walk ω' such that $F' = F \triangle \{\omega, \omega'\}$ is a non-kissing facet.*

3.1.3 Applications to representation theory.

In this section, we give a representation-theoretic interpretation of Theorem 3.1.19. We note that this result was also obtained by T. Brüstle, G. Douville, K. Mousavand, H. Thomas and E. Yıldırım [BDM+17].

We fix a gentle bound quiver (Q, I).

Remark 3.1.20. To any string σ one can associate an indecomposable representation $M(\sigma)$ of (Q, I), called a string module: The arrows in the string describe the action of the linear maps on the basis vectors. Two string modules are isomorphic if and only if the corresponding unoriented strings are equal.

Notation 3.1.21. We extend the notation of the previous remark by letting $M(-v) = P_v[-1]$.

In order to interpret walks on (Q, I) as representations of (Q, I), we thus want to associate a string for (Q, I) to each walk.

Definition 3.1.22. 1. Let σ be a string for (Q, I). A *left cohook* $\searrow$ for σ is a string for $(Q^{\circledast}, I^{\circledast})$ which is maximal of the form $\alpha_r \cdots \alpha_1 \beta^{-1}$ such that $\alpha_r \cdots \alpha_1 \beta^{-1} \sigma$ is a string for $(Q^{❀}, I^{❀})$. A left cohook always exists, and is unique except if σ has length zero in which case σ has two left cohooks (this will not cause trouble in the rest of the definition as these two left cohooks are also right cohooks). *Right cohooks* $\checkmark$, *left hooks* $\nearrow$ and *right hooks* $\nwarrow$ are defined similarly.

2. The *walk* ω_σ associated with a string σ is the walk obtained by adding a left and a right cohook: $\omega_\sigma = \searrow\sigma\checkmark$.

3. The *walk* ω_{-v} associated with a negative simple string $-v$ is the unique walk of shape $\alpha_r \cdots \alpha_1 \varepsilon_v \beta_1^{-1} \cdots \beta_s^{-1}$. The corresponding string module for $(Q^{❀}, I^{❀})$ is the indecomposable injective at v.

Theorem 3.1.23 ([PPP17, BDM+17])**.** *Let (Q, I) be any gentle bound quiver, with blossoming $(Q^{❀}, I^{❀})$ and let $\Lambda = \mathbb{K}Q/I$.*

1. *The map $\omega_? : \mathcal{S}(Q, I)_{\geq -1} \to \mathcal{W}_{\text{bend}}$ is bijective.*

2. *For any almost positive strings $\rho, \sigma \in \mathcal{S}(Q, I)_{\geq -1}$, the walk ω_ρ kisses the walk ω_σ if and only if $\operatorname{Hom}_\Lambda\big(M(\sigma), \tau M(\rho)\big) \neq 0$.*

Notation 3.1.24. In the theorem above, we used of the following convention. If $P[1]$ is a shifted projective, then

- $\operatorname{Hom}_A(M, \tau P[1]) = \operatorname{Hom}_A(P, M)$ for any representation or shifted projective M,
- $\operatorname{Hom}_A(P[1], \tau M) = 0$ for any representation M.

These choices are motivated by the equivalence [AIR14, Lemma 3.4]

$$\operatorname{Hom}_A(M, \tau N) = 0 \Leftrightarrow \operatorname{Hom}_{K^b(\operatorname{proj} A)}(P_N, P_M[1]) = 0,$$

where P_M and P_N are the minimal projective presentations of the representations M and N.

Corollary 3.1.25. *The map $\omega_? : \mathcal{S}(Q, I)_{\geq -1} \to \mathcal{W}_{\text{bend}}$ induces a bijection between support τ-tilting modules over (Q, I) and non-kissing facets. Under this bijection, flips correspond to support τ-tilting mutation.*

Remark 3.1.26.
1. Our proof in [PPP17] is quite long compared to the proof in [BDM+17]. The reason for this is that we first prove a more general result that holds for any string algebra, and then specialise to gentle algebras.

2. Our result for string algebras gives a relatively easy, combinatorial characterisation of the pairs of strings (ρ, σ) such that $\operatorname{Hom}_\Lambda\big(M(\sigma), \tau M(\rho)\big) \neq 0$. Such a result was already known to Florian Eisele, Geoffrey Janssens and Theo Raedschelders [EJR18]. The proof in [EJR18] uses two-term complexes of projectives, whereas we work directly with modules over A. This explains why our combinatorial characterisation is, arguably, quite easier to state.

3.2 Non-kissing complexes are non-crossing complexes.

In the article [PPP17] (see Section 3.1) we have generalised the non-kissing complex from the case of a grid (due to T. McConville) to the case of gentle algebras. Similar combinatorics arise from the work of A. Garver and T. McConville on the non-crossing complex of accordions in a disc. In [PPP18], we generalise the non-crossing complex from the case of a disc to the case of any marked oriented surface. Once thus generalised, the non-kissing and the non-crossing complexes become isomorphic. This fact could not be made mathematically precise without passing to the respective generalisations because very few grids can be described by using accordions on a disc and very few polygon dissections can be described in terms of grids.

3.2.1 Example of a disc.

We begin this section with a short description of polygon dissections and their accordions.

We fix a $4n$-gon $\mathscr{P}$, whose vertices we number from 1 to $4n$ in clockwise order, and a dissection D (i.e. a collection of non-crossing diagonals, a.k.a. partial triangulation) of $\mathscr{P}$. We denote by $\mathcal{V}$ (resp. $\mathcal{V}^*$, B) the set of vertices of the form $4k$ (resp. $4k+2$, $2k+1$). A *D-cell* is (the closure of) a connected component of $\mathscr{P} \setminus \mathrm{D}$. We assume that each D-cell contains at most one vertex in $\mathcal{V}^*$. We call *$\mathcal{V}$-diagonal* a diagonal of $\mathscr{P}$ both of whose endpoints belong to $\mathcal{V}$, and we similarly define *B-diagonals*. We assume that all arcs in the dissection D are $\mathcal{V}$-diagonals. A B-diagonal is called *external* if it is of the form $(4k-1, 4k+1)$ and *internal* otherwise.

Definition 3.2.1. Define the quiver Q_{D} whose vertices are the diagonals of D and whose arrows are the angles formed by two diagonals in D, oriented counterclockwise. The ideal I_{D} of the path algebra $\mathbb{K}Q_{\mathrm{D}}$ is the ideal of relations generated by the paths of length two obtained by composing two arrows corresponding to two different angles in a same D-cell. We let Λ_{D} be the quotient of the path algebra $\mathbb{K}Q_{\mathrm{D}}$ by the ideal I_{D}.

Remark 3.2.2. Let Λ_{D} be the algebra of Definition 3.2.1.

1. The algebra Λ_{D} is the endomorphism algebra of a rigid object (associated with D) in the cluster category $\mathcal{C}_{\mathrm{A}_{4n-3}}$.

2. The algebra Λ_{D} is a gentle algebra of finite representation type.

3. The algebras of the form Λ_{D} are precisely the *tiling algebras* of Raquel Coelho Simões and Mark Parsons [CSP17], which appear in the article [GG18].

An example of the gentle bound quiver of a dissection is given in Figure 3.4, along with its blossoming bound quiver.

Definition 3.2.3. (slightly modified version of [MP19]) A D-accordion is a B-diagonal that crosses a connected subset of the diagonals in D.

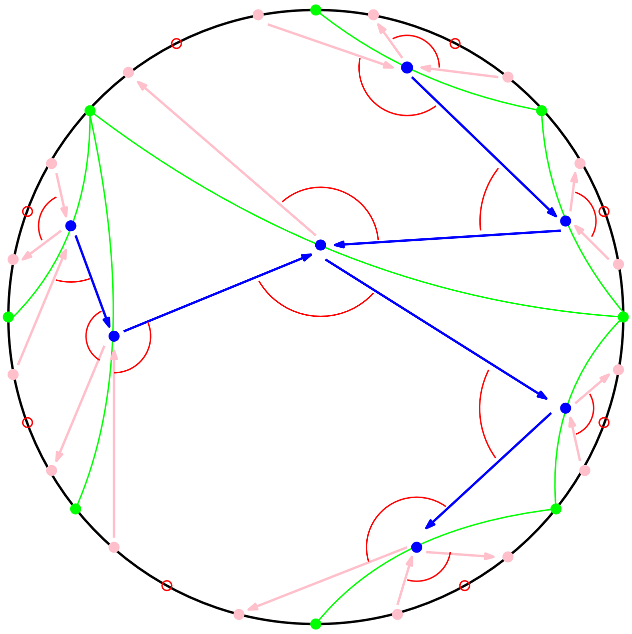


Figure 3.4: The gentle bound quiver of a polygon dissection (green), and its blossoming. The quiver is blue, and blossoming arrows and vertices are pink. All relations are red.

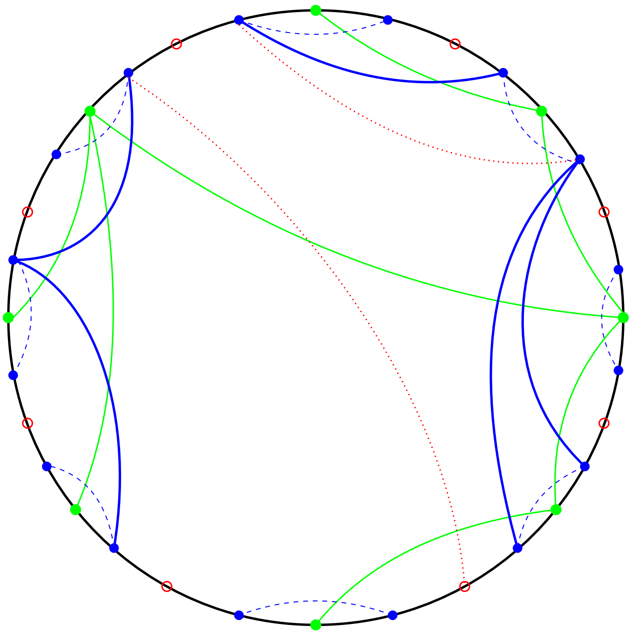


Figure 3.5: A polygon dissection (green), the external accordions (dashed blue), some accordions (plain blue) and some non-accordions (dotted red).

Remark 3.2.4. We note that if D is a triangulation, then any B-diagonal is a D-accordion.

We give several examples, and non-examples, of accordions in Figure 3.5

Definition 3.2.5. A *non-crossing facet* is a maximal set of pairwise non-crossing D-accordions.

Note that the external B-diagonals are D-accordions and do not cross any D-accordion; they thus belong to any non-crossing facet. The well-behaved combinatorics of triangulations and their flips generalise to the case of D-accordions.

Theorem 3.2.6. [GM18, Corollaries 3.6 and 3.9] *Let* D *be a dissection of* $\mathscr{P}$*. Then*

1. *Any non-crossing facet has the same cardinality: The number of internal* D*-accordions in each facet is the number of diagonals in the dissection* D*.*

2. *Let* F *be a non-crossing facet, and let* $\gamma \in F$ *be an internal* D*-accordion. Then there is a unique other internal accordion* γ' *such that* $F' = F \triangle \{\gamma, \gamma'\}$ *is a non-crossing facet.*

Definition 3.2.7. The facet F' above is called the *flip* of F at γ.

3.2.2 Dissections and accordions.

In order to relate non-crossing complexes and non-kissing complexes, we first generalise the non-crossing complex of accordions on a disc to the case of marked orientable surfaces. Dissections on surfaces are simply collections of non-crossing arcs on the surface ; but the definition of an accordion has to be slightly modified in order to be suitable for more general surfaces than discs. We note that punctures are allowed.

Definition 3.2.8. A *marked surface* $(\mathcal{S}, M)$ is an orientable surface $\mathcal{S}$ with boundaries, together with a set M of marked points which can be on the boundary of $\mathcal{S}$ or not. For $V \subset \mathcal{S}$,

(i) a *V-arc* on $(\mathcal{S}, M)$ is a curve on $\mathcal{S}$ connecting two points of V and whose interior is disjoint from M and from the boundary of $\mathcal{S}$.

(ii) a *V-curve* on $(\mathcal{S}, M)$ is a curve on $\mathcal{S}$ which at each end either reaches a point of V or infinitely circles around and finally reaches a puncture of M, and whose interior is disjoint from M and the boundary of $\mathcal{S}$.

As usual, curves and arcs are considered up to homotopy relative to their endpoints in $\mathcal{S} \smallsetminus M$, and curves homotopic to a boundary are not allowed.

Definition 3.2.9. A *dissection* of $(\mathcal{S}, M)$ is a collection D of pairwise non-crossing arcs on $(\mathcal{S}, M)$. The *edges* of D are its arcs together with the boundary arcs of $(\mathcal{S}, M)$. The *faces* of D are the connected components of the complement of the union of the edges of D in the surface $\mathcal{S}$. We denote by $\mathcal{V}(\mathrm{D})$, $\mathcal{E}(\mathrm{D})$ and $\mathcal{F}(\mathrm{D})$ the sets of vertices, edges and faces of D respectively. We always assume that the dissection D is *cellular*, i.e. that all its faces are topological disks. For $V \subseteq M$, a *V-dissection* is a dissection with only V-arcs.

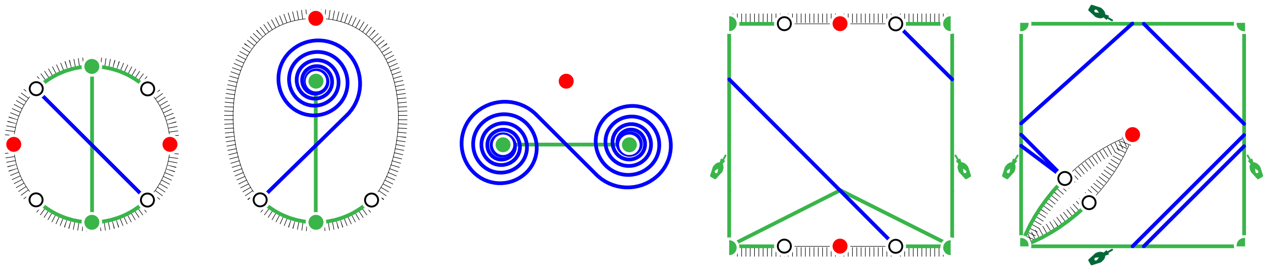


Figure 3.6: Some D-accordions (in blue) for the dissections (in green) of a disc, a punctured disc, a surface with two punctures, a cylinder and a torus.

Definition 3.2.10. A cellular V-dissection D of a marked surface $(\mathcal{S}, V \sqcup V^*)$ is called *dualizable* if each face of D contains exactly one point of V^* (in particular, at most one boundary edge). In that case, there is a unique *dual* cellular V^*-dissection D^*.

Definition 3.2.11. We consider a set B of points on the boundary of the surface $\mathcal{S}$ such that B and $V \cup V^*$ alternate along the boundary of $\mathcal{S}$. The points of B are called the *blossom points*. We say that a B-curve is *external* if it is homotopic to a boundary arc of $\mathcal{S} \smallsetminus B$, and *internal* otherwise. We note that no B-curve can cross an external B-curve. See Figure 3.5, and Figure 3.6 where the blossom points appear as white hollow vertices.

The following definition generalises the one of [MP19] for the case where $\bar{\mathcal{S}}$ is a disk. A very similar definition appears in [BCS18] in a slightly different context under the name “permissible arc”.

Definition 3.2.12. Consider two dual cellular dissections D and D^* of $(\mathcal{S}, M)$, where $M = V \sqcup V^*$, and let B be the set of blossom points as above. A D-*accordion* is a B-curve α of $(\mathcal{S}, M)$ such that whenever α meets a face f of D,

(i) it enters crossing an edge a of f and leaves crossing an edge b of f (in other words, α is not allowed to circle around f^* when f^* is a puncture),

(ii) the two edges a and b of f crossed by α are consecutive along the boundary of f,

(iii) α, a and b bound a disk inside f that does not contain f^*.

By convention, we also consider that the punctures of V are D-accordions that are considered external. If we were working on the universal cover of the surface, the D-accordion associated to a puncture would be the infinite line crossing all (infinitely many) arcs attached to the puncture.

Definition 3.2.13. The D-*accordion complex* $\mathcal{K}_{\mathrm{acc}}(\mathrm{D})$ is the simplicial complex whose faces are the collections of pairwise non-crossing D-accordions. We note that, by definition, self-crossing accordions never appear in $\mathcal{K}_{\mathrm{acc}}(\mathrm{D})$ and that the external accordions appear in all facets of $\mathcal{K}_{\mathrm{acc}}(\mathrm{D})$.

Definition 3.2.14. We call *non-crossing complex* of the pair of dual cellular dissections $(\mathrm{D}, \mathrm{D}^*)$ the simplicial complex $\mathcal{K}_{\mathrm{nc}}(\mathrm{D}, \mathrm{D}^*) := \mathcal{K}_{\mathrm{acc}}(\mathrm{D})$.

The appearance of the dual cellular dissection D^* in the non-crossing complex $\mathcal{K}_{\mathrm{nc}}(\mathrm{D}, \mathrm{D}^*)$ is motivated by the fact that D-accordions can alternatively be defined as D^*-slaloms (see [PPP18]).

3.2.3 Locally gentle algebras vs surface dissections.

In order to compare the non-kissing complex and the non-crossing complex, we first relate dissections of surfaces with gentle algebras: We associate a gentle bound quiver to any surface dissection, and conversely, we use the blossoming bound quiver to construct a surface with a pair of dual dissections associated with any gentle bound quiver.

The idea of associating an algebra to a surface dissection is not new. Instances of such constructions appear in the theory of cluster algebras for triangulations [CCS06, ABCJP10, LF09] and for more general dissections [Dem16, Dem17, DRS12]. In many cases, the algebras obtained are gentle. It has been shown in [BCS18] that any gentle algebra can be obtained from some surface dissection, and that the module category of the algebra has a combinatorial description in terms of curves on the surface.

Conversely, the construction of a surface associated with a gentle algebra has appeared in [OPS18] where the surface is shown to model the derived category of the gentle algebra. We give a different construction of the same surface in [PPP18], which is obtained by "glueing" triangles to the arrows of the blossoming quiver, as explained below. Our construction, which is expected to model the category of two-term complexes of projectives, has the advantage that it easily yields two dual dissections on the surface at the same time (the dissection and dual lamination of [OPS18]). Note that our dissections are always cellular, while those in [BCS18] can be arbitrary. Remarkably, gentle algebras and surfaces were linked recently in [HKK17, LP18], where the Fukaya category of the surface is shown to be equivalent to the bounded derived category of the associated gentle algebra.

Because punctures are allowed on the combinatorial side (the non-crossing complex), we are led to consider locally-gentle algebras. Those are the algebras obtained by dropping the assumption that the ideal is admissible in the definition of a gentle algebra. In other words, a locally-gentle algebra is a gentle algebra that is not necessarily finite-dimensional. Locally-gentle algebras are simply called gentle algebras by several authors, e.g. [Sch99].

Definition 3.2.15. A *locally-gentle* algebra is the algebra of a bound quiver (Q, I) that satisfies all the assumptions for being gentle (Definition 3.1.12) but admissibility of I.

Let D and D^* be two dual cellular dissections of a marked surface $(\mathcal{S}, V \sqcup V^*)$, and let B be the set of blossom points.

Definition 3.2.16. The bound quiver of the dissection D is the bound quiver $(Q_{\mathrm{D}}, I_{\mathrm{D}})$ defined as follows:

(i) the set of vertices of Q_{D} is the set of edges of D;

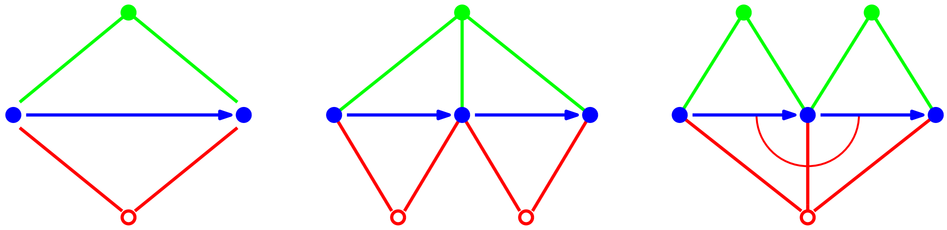


Figure 3.7: Constructing a surface from a locally-gentle bound quiver (Definition 3.2.20).

(ii) there is an arrow from a to b for each common endpoint v of a and b such that b comes immediately after a in the counterclockwise order around v;

(iii) the ideal I_{D} is generated by the paths of length two in Q_{D} obtained by composing arrows which correspond to triples of consecutive edges in a face of D.

The bound quiver of the dissection D^* is the bound quiver $(Q_{\mathrm{D}^*}, I_{\mathrm{D}^*})$ defined by replacing D by D^* in the above.

Remark 3.2.17. The blossoming bound quiver $(Q_{\mathrm{D}}^{\text{❀}}, I_{\mathrm{D}}^{\text{❀}})$ of the bound quiver $(Q_{\mathrm{D}}, I_{\mathrm{D}})$ is obtained with the same procedure by considering additional blossom vertices along the boundary of the surface.

Lemma 3.2.18. *The bound quiver $(Q, I)_{\mathrm{D}} = (Q_{\mathrm{D}}, I_{\mathrm{D}})$ is a locally gentle bound quiver.*

Remark 3.2.19. It easily follows from [BH08] that the bound quiver $(Q_{\mathrm{D}^*}, I_{\mathrm{D}^*})$ is the (ungraded) Koszul dual of $(Q_{\mathrm{D}}, I_{\mathrm{D}})$. In this specific setup, this can be reformulated as follows: $Q_{\mathrm{D}^*} = Q_{\mathrm{D}}^{\mathrm{op}}$ and, for any pair of composable arrows $\alpha, \beta \in (Q_{\mathrm{D}})_1$, $\beta\alpha \in I_{\mathrm{D}^*}$ if and only if $\alpha\beta \notin I_{\mathrm{D}}$. In particular, the Koszul dual of a finite-dimensional gentle algebra may be infinite-dimensional.

Definition 3.2.20. The surface $\mathcal{S}_{(Q,I)}$ of a locally-gentle bound quiver (Q, I) is the surface obtained as follows:

1. Replace (Q, I) by its blossoming $(Q^{\text{❀}}, I^{\text{❀}})$.
2. Glue filled-in triangles on each side of each arrow, one with two green edges, the other one with two red edges, as shown in Figure 3.7 (left).
3. For any pair of composable arrows $\alpha, \beta \in Q_1^{\text{❀}}$, glue the associated green triangles along their consecutive edges if $\alpha\beta \notin I^{\text{❀}}$ (Figure 3.7, middle).
4. For any pair of composable arrows $\alpha, \beta \in Q_1^{\text{❀}}$, glue the associated red triangles along their consecutive edges if $\alpha\beta \in I^{\text{❀}}$ (Figure 3.7, right).

The advantage of Definition 3.2.20 is that it automatically endows $\mathcal{S}_{(Q,I)}$ with two disjoint sets $V_{(Q,I)}$ and $V^*_{(Q,I)}$ of marked points and two dual cellular dissections $\mathrm{D}_{(Q,I)}$ and $\mathrm{D}^*_{(Q,I)}$ defined as follows.

Definition 3.2.21. The surface $\mathcal{S}_{(Q,I)}$ is endowed with

- the set $V_{(Q,I)}$ of green points in Figure 3.7, after the identifications given by step 3.,
- the $V_{(Q,I)}$-dissection $\mathrm{D}_{(Q,I)}$ given by all green edges, after the identifications given by step 3.

The set $V^*_{(Q,I)}$ and the $V^*_{(Q,I)}$-dissection $\mathrm{D}^*_{(Q,I)}$ are defined similarly by using the red vertices and the red edges.

Proposition 3.2.22. *Let (Q,I) be a locally-gentle bound quiver. Then the dissections $\mathrm{D}_{(Q,I)}$ and $\mathrm{D}^*_{(Q,I)}$ are cellular and dual to each other.*

Theorem 3.2.23. *Up to isomorphism, the constructions of Definitions 3.2.16 and 3.2.20 are inverse to each other. They induce a bijection between the set of isomorphism classes of locally gentle bound quivers and the set of homeomorphism classes of marked surfaces with a pair of dual cellular dissections.*

Remark 3.2.24. Fix a locally-gentle bound quiver (Q,I) and let $(Q,I)^!$ be its Koszul dual (see Remark 3.2.19). The following observations are useful for the computation of examples.

(i) The set $V_{(Q,I)}$ has one vertex for each straight walk in (Q,I) (equivalently, for each maximal path in (Q,I)). Finite straight walks yield vertices on the boundary of $\mathcal{S}_{(Q,I)}$, while infinite cyclic straight walks in (Q,I) yield punctures of $\mathcal{S}_{(Q,I)}$ in $V_{(Q,I)}$. We denote by p the number of infinite cyclic straight walks in (Q,I).

(ii) The dissection $\mathrm{D}_{(Q,I)}$ has one edge, denoted by $\varepsilon(a)$, for each vertex $a \in Q_0$, obtained by concatenation of the two (after the identifications of step 3. in Definition 3.2.20) green edges that contain a.

(iii) The dissection $\mathrm{D}_{(Q,I)}$ has one ℓ-cell for each straight walk of length ℓ in $(Q,I)^!$.

(iv) Similar statements hold dually for $V^*_{(Q,I)}$ and $\mathrm{D}^*_{(Q,I)}$, and the notations p^* and $\varepsilon^*(a)$ are defined similarly.

(v) The number of punctures of $\mathcal{S}_{(Q,I)}$ is the number $p + p^*$ of infinite straight walks in (Q,I) and in $(Q,I)^!$.

(vi) The number b of boundary components of $\mathcal{S}_{(Q,I)}$ can be computed as the number of orbits of blossoming vertices. Here, the orbit of a blossoming vertex v is computed as follows: Assuming that v is a source, follow the maximal path in $Q^{❀}$ starting at v to obtain a new blossoming vertex v'. Now follow the maximal path in $(Q^{❀})^!$ starting at v' (equivalently, follow the maximal antipath in $Q^{❀}$ ending in v') to obtain the blossoing vertex v''. Iterating this procedure gives all vertices in the orbit of v.

(vii) The genus of the surface $\mathcal{S}_{(Q,I)}$ is

$$g = \frac{|Q_1| - |Q_0| - b - p - p^* + 2}{2},$$

where b is the number of boundary components (see above for a way to compute b) and $p+p^*$ the number of punctures (*i.e.* infinite straight walks in (Q,I) and in $(Q,I)^!$).

Remark 3.2.25. Using the blossoming bound quiver, the Avella-Alaminos–Geiss invariant translates into an invariant that is easily read from the surface. Claire Amiot, Pierre-Guy Plamondon and Sibylle Schroll used this point of view in order to extend the AAG-invariant into a complete derived invariant. See the beautiful [APS19].

3.2.4 Non-kissing vs non-crossing.

We are now ready to compare the combinatorics of walks and those of accordions. Fix a locally-gentle bound quiver (Q,I) and let $\mathcal{S}_{(Q,I)}$ be the associated marked surface, endowed with the pair of dual cellular dissections $(\mathrm{D}_{(Q,I)}, \mathrm{D}^*_{(Q,I)})$ as defined in Section 3.2.3. With each walk on (Q,I), we associate a curve on $\mathcal{S}_{(Q,I)}$:

Definition 3.2.26. Let ω be a walk on (Q,I). The associated *curve*, $\gamma(\omega)$, is obtained by first drawing the walk on the quiver $Q^{❀}$, and then by embedding the quiver on the surface $\mathcal{S}_{(Q,I)}$.

Lemma 3.2.27. *Let ω be a walk on (Q,I). Then $\gamma(\omega)$ is a $\mathrm{D}_{(Q,I)}$-accordion.*

Lemma 3.2.28. *Two undirected walks ω_1 and ω_2 on (Q,I) are non-kissing if and only if the corresponding $\mathrm{D}_{(Q,I)}$-accordions $\gamma(\omega_1)$ and $\gamma(\omega_2)$ are non-crossing on $\mathcal{S}_{(Q,I)}$.*

Theorem 3.2.29. *The non-kissing and non-crossing complexes are isomorphic:*

- *for any locally gentle bound quiver (Q,I), the non-kissing complex $\mathcal{NK}(Q,I)$ is isomorphic to the non-crossing complex $\mathcal{K}_{\mathrm{nc}}(\mathrm{D}_{(Q,I)}, \mathrm{D}^*_{(Q,I)})$,*
- *for any pair of dual cellular dissections D, D^* of an oriented surface, the non-crossing complex $\mathcal{K}_{\mathrm{nc}}(\mathrm{D}, \mathrm{D}^*)$ is isomorphic to the non-kissing complex $\mathcal{NK}(Q_{\mathrm{D}}, I_{\mathrm{D}})$.*

Remark 3.2.30. For Theorem 3.2.29 to make sense, one has to first generalise the non-kissing complex to the case of locally-gentle algebras. This is done in [PPP18], where we also give purely combinatorial proofs that those complexes (more precisely: their reduced versions) are *pure and thin*, i.e. that all of their facets have the same cardinality, and that there are well-defined notions of flips.

Remark 3.2.31. If the bound quiver (Q,I) is gentle, then the non-kissing complex $\mathcal{NK}(Q,I)$, and hence the non-crossing complex $\mathcal{K}_{\mathrm{nc}}(\mathrm{D}_{(Q,I)}, \mathrm{D}^*_{(Q,I)})$, is isomorphic to the support τ-tilting complex of (Q,I) (see Section 3.1.3), and flips encode support τ-tilting mutations. The fact that Theorem 3.1.19 generalises to locally-gentle algebras suggests that some generalisation of τ-tilting theory might be defined for infinite-dimensional locally-gentle algebras (or their completions).

Chapter 4

Extriangulated categories.

Extriangulated categories, recently introduced in [NP19], axiomatize extension-closed subcategories of triangulated categories in a (moderately) similar way that Quillen's exact categories axiomatize extension-closed subcategories of abelian categories. They appear in representation theory in relation with cotorsion pairs [ZH16, LN17, Liu17], with Auslander–Reiten theory [INP18], with cluster algebras, mutations, or cluster-tilting theory [CZZ16, Pre17, ZZ18, LZ18a, LZ18b, PPPP19, ZZ19, Zho19], with Gorenstein-projective objects [LZ19a, HZZ19b, HZZ19a], with τ-tilting theory [LZ19b], with Cohen–Macaulay dg-modules in the remarkable [Jin18]. We also note the generalization, called n-exangulated categories [HLN17], to a version suited for higher homological algebra.

4.1 The axioms for extriangulated categories.

An extriangulated category is the data of an additive category $\mathscr{C}$, an additive bifunctor $\mathbb{E} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab$ modelling the Ext^1-bifunctor, and an additive realization $\mathfrak{s}$ sending each element $\delta \in \mathbb{E}(Z, X)$ to some (equivalence classe of) diagram $X \to Y \to Z$ modelling the short exact sequences or triangles. Some axioms, inspired from the case of extension-closed subcategories of triangulated categories have to be satisfied.

4.1.1 Definitions and first properties.

More specifically: fix an additive category $\mathscr{C}$, and an additive bifunctor $\mathbb{E} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab$.

Definition 4.1.1. For any $X, Z \in \mathscr{C}$, an element $\delta \in \mathbb{E}(Z, X)$ is called an *$\mathbb{E}$-extension*. A *split* $\mathbb{E}$-extension is a zero element $0 \in \mathbb{E}(Z, X)$, for some objects $X, Z \in \mathscr{C}$. For any two $\mathbb{E}$-extensions $\delta \in \mathbb{E}(Z, X)$, $\delta' \in \mathbb{E}(Z', X')$, the additivity of $\mathscr{C}$, $\mathbb{E}$ permits to define the $\mathbb{E}$-extension

$$\delta \oplus \delta' \in \mathbb{E}(Z \oplus Z', X \oplus X').$$

Remark 4.1.2. Let $\delta \in \mathbb{E}(Z, X)$ be an $\mathbb{E}$-extension. By functoriality, any morphisms $f \in \mathscr{C}(X, X')$ and $h \in \mathscr{C}(Z', Z)$ induce $\mathbb{E}$-extensions $\mathbb{E}(Z, f)(\delta) \in \mathbb{E}(Z, X')$ and $\mathbb{E}(h, X)(\delta) \in$

$\mathbb{E}(Z', X)$. For short, we write $f_*\delta$ and $h^*\delta$ instead. Using those notations, we have, in $\mathbb{E}(Z', X')$

$$\mathbb{E}(h, f)(\delta) = h^* f_* \delta = f_* h^* \delta.$$

Definition 4.1.3. A *morphism* $(f, h) : \delta \to \delta'$ of $\mathbb{E}$-extensions $\delta \in \mathbb{E}(Z, X)$, $\delta' \in \mathbb{E}(Z', X')$ is a pair of morphisms $f \in \mathscr{C}(X, X')$ and $h \in \mathscr{C}(Z, Z')$ in $\mathscr{C}$, such that $f_*\delta = h^*\delta'$.

Definition 4.1.4. Let $X, Z \in \mathscr{C}$ be any two objects. Two sequences of morphisms in $\mathscr{C}$

$$X \xrightarrow{x} Y \xrightarrow{y} Z \quad \text{and} \quad X \xrightarrow{x'} Y' \xrightarrow{y'} Z$$

are said to be *equivalent* if there exists an isomorphism $g \in \mathscr{C}(Y, Y')$ such that the following diagram commutes.

$$\begin{tikzcd} & Y \arrow[dr, "y"] \arrow[dd, "g", "\cong"'] & \\ X \arrow[ur, "x"] \arrow[dr, "x'"'] & & Z \\ & Y' \arrow[ur, "y'"'] & \end{tikzcd}$$

The equivalence class of $X \xrightarrow{x} Y \xrightarrow{y} Z$ is denoted by $[X \xrightarrow{x} Y \xrightarrow{y} Z]$.

Notation 4.1.5. For any $X, Y, Z, A, B, C \in \mathscr{C}$, and any $[X \xrightarrow{x} Y \xrightarrow{y} Z]$, $[A \xrightarrow{a} B \xrightarrow{b} C]$, we let

$$0 = [X \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} X \oplus Y \xrightarrow{\left[\begin{smallmatrix}0 & 1\end{smallmatrix}\right]} Y]$$

and

$$[X \xrightarrow{x} Y \xrightarrow{y} Z] \oplus [A \xrightarrow{a} B \xrightarrow{b} C] = [X \oplus A \xrightarrow{\left[\begin{smallmatrix}x & 0\\0 & a\end{smallmatrix}\right]} Y \oplus B \xrightarrow{\left[\begin{smallmatrix}y & 0\\0 & b\end{smallmatrix}\right]} Z \oplus C].$$

Definition 4.1.6. An *additive realization* $\mathfrak{s}$ is a correspondence associating, with $\mathbb{E}$-extension $\delta \in \mathbb{E}(Z, X)$, an equivalence class $\mathfrak{s}(\delta) = [X \xrightarrow{x} Y \xrightarrow{y} Z]$ and satisfying the following condition:

$(*)$ Let $\delta \in \mathbb{E}(Z, X)$ and $\delta' \in \mathbb{E}(Z', X')$ be any pair of $\mathbb{E}$-extensions, with

$$\mathfrak{s}(\delta) = [X \xrightarrow{x} Y \xrightarrow{y} Z] \text{ and } \mathfrak{s}(\delta') = [X' \xrightarrow{x'} Y' \xrightarrow{y'} Z'].$$

Then, for any morphism $(f, h) : \delta \to \delta'$, there exists $g \in \mathscr{C}(Y, Y')$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[r, "y"] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowright"] & Z \arrow[d, "h"] \\ X' \arrow[r, "x'"'] & Y' \arrow[r, "y'"'] & Z' \end{tikzcd}$$

The sequence $X \xrightarrow{x} Y \xrightarrow{y} Z$ is said to *realize* δ if $\mathfrak{s}(\delta) = [X \xrightarrow{x} Y \xrightarrow{y} Z]$, and the triple (f, g, h) is said to realize (f, h) if the diagram in $(*)$ commutes.

Definition 4.1.7. A realization of $\mathbb{E}$ is called an *additive realization* if the following conditions are satisfied:

1. For any $X, Z \in \mathscr{C}$, the realization of the split $\mathbb{E}$-extension $0 \in \mathbb{E}(Z, X)$ is given by $\mathfrak{s}(0) = 0$.

2. For any two $\mathbb{E}$-extensions $\delta \in \mathbb{E}(Z, X)$ and $\delta' \in \mathbb{E}(Z', X')$, the realization of $\delta \oplus \delta'$ is given by $\mathfrak{s}(\delta \oplus \delta') = \mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta')$.

Definition 4.1.8. ([NP19, Definition 2.12]) A triple $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is called an *extriangulated category* if the following holds:

(ET1) $\mathbb{E} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab$ is an additive bifunctor;

(ET2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$;

(ET3) Let $\delta \in \mathbb{E}(Z, X)$ and $\delta' \in \mathbb{E}(Z', X')$ be $\mathbb{E}$-extensions, respectively realized by $X \xrightarrow{x} Y \xrightarrow{y} Z$ and $X' \xrightarrow{x'} Y' \xrightarrow{y'} Z'$. Then, for any commutative square

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[r, "y"] \arrow[d, "g"] & Z \\ X' \arrow[r, "x'"'] & Y' \arrow[r, "y'"'] & Z' \end{tikzcd}$$

in $\mathscr{C}$, there exists a morphism $(f, h) : \delta \to \delta'$ satisfying $h \circ y = y' \circ g$.

(ET3)$^{\mathrm{op}}$ Dual of (ET3).

(ET4) Let $\delta \in \mathbb{E}(Z', X)$ and $\delta' \in \mathbb{E}(X', Y)$ be $\mathbb{E}$-extensions realized respectively by

$$X \xrightarrow{f} Y \xrightarrow{f'} Z' \quad \text{and} \quad Y \xrightarrow{g} Z \xrightarrow{g'} X'.$$

Then there exist an object $Y' \in \mathscr{C}$, a commutative diagram in $\mathscr{C}$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[r, "f'"] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowright"] & Z' \arrow[r, dashed, "\delta"] \arrow[d, "d"] & {} \\ X \arrow[r, "h"] & Z \arrow[r, "h'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\circlearrowright"] & Y' \arrow[r, dashed, "\delta''"] \arrow[d, "e"] & {} \\ & X' \arrow[r, equal] \arrow[d, dashed, "\delta'"] & X' \arrow[d, dashed, "f'_*\delta'"] & \\ & {} & {} & \end{tikzcd}$$

and an $\mathbb{E}$-extension $\delta'' \in \mathbb{E}(Y', X)$ realized by $X \xrightarrow{h} Z \xrightarrow{h'} Y'$, which satisfy the following compatibilities.

(i) $Z' \xrightarrow{d} Y' \xrightarrow{e} X'$ realizes $f'_*\delta'$,

(ii) $d^*\delta'' = \delta$,

(iii) $f_*\delta'' = e^*\delta'$.

(ET4)$^{\mathrm{op}}$ Dual of (ET4).

We use the following terminology.

Notation 4.1.9. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category.

1. A sequence $X \xrightarrow{x} Y \xrightarrow{y} Z$ is called a conflation if it realizes some $\mathbb{E}$-extension in $\mathbb{E}(Z,X)$. In which case the morphism $X \xrightarrow{x} Y$ is called an inflation, written $X \rightarrowtail Y$, and the morphism $Y \xrightarrow{y} Z$ is called a deflation, witten $Y \twoheadrightarrow Z$.

2. An extriangle is a diagram $X \overset{x}{\rightarrowtail} Y \overset{y}{\twoheadrightarrow} Z \overset{\delta}{\dashrightarrow}$ where $X \overset{x}{\rightarrowtail} Y \overset{y}{\twoheadrightarrow} Z$ is a conflation realizing the $\mathbb{E}$-extension $\delta \in \mathbb{E}(Z,X)$.

3. Similarly, we call morphism of extriangles any diagram

$$\begin{tikzcd} X \arrow[r, tail, "x"] \arrow[d, "f"'] & Y \arrow[r, two heads, "y"] \arrow[d, "g"] & Z \arrow[r, dashed, "\delta"] \arrow[d, "h"] & {} \\ X' \arrow[r, tail, "x'"'] & Y' \arrow[r, two heads, "y'"'] & Z' \arrow[r, dashed, "\delta'"'] & {} \end{tikzcd}$$

where $(f,h) : \delta \to \delta'$ is a morphism of $\mathbb{E}$-extensions realized by (f,g,h).

The axioms above ensure that any extriangle induce long-ish exact sequences after application of some covariant or contravariant Hom-functor. In particular in any conflation, the inflation is a weak kernel of the deflation, and the deflation is a weak cokernel of the inflation.

Proposition 4.1.10. *Assume that $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category, and let $X \xrightarrow{x} Y \xrightarrow{y} Z \overset{\delta}{\dashrightarrow}$ be an extriangle. Then the following sequences of natural transformations are exact:*

$$\mathscr{C}(Z,-) \xrightarrow{-\circ y} \mathscr{C}(Y,-) \xrightarrow{-\circ x} \mathscr{C}(X,-) \xrightarrow{\delta^\sharp} \mathbb{E}(Z,-) \xrightarrow{y^*} \mathbb{E}(Y,-) \xrightarrow{x^*} \mathbb{E}(X,-),$$

$$\mathscr{C}(-,X) \xrightarrow{x\circ -} \mathscr{C}(-,Y) \xrightarrow{y\circ -} \mathscr{C}(-,Z) \xrightarrow{\delta_\sharp} \mathbb{E}(-,X) \xrightarrow{x_*} \mathbb{E}(-,Y) \xrightarrow{y_*} \mathbb{E}(-,Z),$$

where $\delta^\sharp(f) = f_*\delta$ and $\delta_\sharp(g) = g^*\delta$.

Remark 4.1.11. Any variant of the axiom (ET4) that would hold in an extension-closed subcategory of a triangulated category by applying the octahedron axiom also holds in any extriangulated category. See [NP19, Section 3.2] for more details.

Remark 4.1.12. Based on Andrew Hubery's notes on the octahedral axiom, Owen Garnier has studied, in his Master 1 thesis, several equivalent versions of the axiom (ET4).

4.1.2 Relation with exact or triangulated categories.

We claimed that extriangulated categories generalise both exact and triangulated categories, and that they axiomatize extension-closed subcategories of triangulated categories. Let us now justify those claims.

Let $\mathscr{C}$ be an additive category equipped with an equivalence $[1]\colon \mathscr{C} \xrightarrow{\simeq} \mathscr{C}$, and let $\mathbb{E}^1\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab$ be the bifunctor defined by $\mathbb{E}^1 = \mathrm{Ext}^1(-,-) = \mathscr{C}(-,-[1])$.

Lemma 4.1.13. *Assume that $\mathfrak{s}$ is an additive realization such that $(\mathscr{C}, \mathbb{E}^1, \mathfrak{s})$ is extriangulated. Then for any $A \in \mathscr{C}$, $A \to 0 \to A[1] \overset{\mathrm{id}_{A[1]}}{\dashrightarrow}$ is an extriangle.*

Proposition 4.1.14. *We have the following.*

(1) Assume that $\mathscr{C}$ is a triangulated category with the shift functor $[1]$. For any $\delta \in \mathbb{E}^1(C,A) = \mathscr{C}(C, A[1])$, take a distinguished triangle

$$A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} A[1]$$

and let $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$. Then $(\mathscr{C}, \mathbb{E}^1, \mathfrak{s})$ is an extriangulated category.

(2) Conversely, assume that $\mathfrak{s}$ is an additive realization such that $(\mathscr{C}, \mathbb{E}^1, \mathfrak{s})$ is extriangulated. Define that $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} A[1]$ is a distinguished triangle if and only if it satisfies $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$. With this class of distinguished triangles, $\mathscr{C}$ becomes a triangulated category.

Remark 4.1.15. By the construction above, the triangles of a triangulated category $\mathscr{C}$ are precisely the extriangles for the corresponding extriangulated structure.

Let now $\mathscr{E}$ be an exact category. Assume that one of the following three conditions hold:

(i) The category $\mathscr{E}$ is skeletally small.

(ii) The exact category $\mathscr{E}$ has enough projectives.

(iii) The exact category $\mathscr{E}$ has enough injectives.

Then, for any pair of objects $A, C \in \mathscr{E}$, the class $\mathrm{Ext}^1_{\mathscr{E}}(C,A)$, of equivalence classes of conflations, becomes a set. We thus obtain a biadditive functor $\mathrm{Ext}^1_{\mathscr{E}}\colon \mathscr{E}^{\mathrm{op}} \times \mathscr{E} \to Ab$. Its functorial structure is given as follows:

- For any $\delta = [A \xrightarrow{x} B \xrightarrow{y} C] \in \mathrm{Ext}^1(C,A)$ and any $a \in \mathscr{E}(A,A')$, take a pushout in $\mathscr{E}$, to obtain a morphism of short exact sequences

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] \arrow[dr, phantom, "\mathrm{PO}"] & B \arrow[r, "y"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, equal] \\ A' \arrow[r, "m"'] & M \arrow[r, "e"'] & C \end{tikzcd}.$$

This gives $\mathrm{Ext}^1_{\mathscr{E}}(C,a)(\delta) = a_*\delta = [A' \xrightarrow{m} M \xrightarrow{e} C]$.

- For any $c \in \mathscr{E}(C', C)$, the map $\operatorname{Ext}^1_{\mathscr{E}}(c, A) = c^* \colon \operatorname{Ext}^1_{\mathscr{E}}(C, A) \to \operatorname{Ext}^1_{\mathscr{E}}(C', A)$ is defined dually by using pullbacks.

Recall that the zero element in $\operatorname{Ext}^1_{\mathscr{E}}(C, A)$ is given by the split short exact sequence

$$0 = [A \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} A \oplus C \xrightarrow{[0\ 1]} C].$$

For any pair $\delta_1 = [A \xrightarrow{x_1} B_1 \xrightarrow{y_1} C], \delta_2 = [A \xrightarrow{x_2} B_2 \xrightarrow{y_2} C] \in \operatorname{Ext}^1_{\mathscr{E}}(C, A)$, its sum $\delta_1 + \delta_2$ is given by the Baer sum

$$\Delta_C^*(\nabla_A)_*(\delta_1 \oplus \delta_2) = \Delta_C^*(\nabla_A)_*([A \oplus A \xrightarrow{x_1 \oplus x_2} B_1 \oplus B_2 \xrightarrow{y_1 \oplus y_2} C \oplus C]).$$

Proposition 4.1.16. *Define the realization $\mathfrak{s}(\delta)$ of $\delta = [A \xrightarrow{x} B \xrightarrow{y} C]$ to be δ itself. Then $(\mathscr{E}, \operatorname{Ext}^1_{\mathscr{E}}, \mathfrak{s})$ is extriangulated.*

In order to prove this result, the following fact is useful.

Lemma 4.1.17. ([Büh10, Proposition 3.1]) *For any morphism, $(*)$ below, of short exact sequences in $\mathscr{E}$, there exists a commutative diagram $(**)$ whose middle row is also a short exact sequence, the upper-left square is a pushout, and the lower-right square is a pullback. In other words, $a_*[A \xrightarrow{x} B \xrightarrow{y} C] = c^*[A' \xrightarrow{x'} B' \xrightarrow{y'} C']$.*

$$(*)\quad \begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[r, "y"] \arrow[d, "b"] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "c"] \\ A' \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C' \end{tikzcd} \qquad (**)\quad \begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"'] \arrow[dr, phantom, "\text{PO}"] & B \arrow[r, "y"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, equal] \\ A' \arrow[r, "m"'] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & {}^{\exists}M \arrow[r, "e"'] \arrow[d] \arrow[dr, phantom, "\text{PB}"] & C \arrow[d, "c"] \\ A' \arrow[r, "x'"'] & B' \arrow[r, "y'"'] & C' \end{tikzcd}$$

Remark 4.1.18. Some converse to Proposition 4.1.16 holds: If $(\mathscr{E}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category all of whose inflations are monomorphisms and all of whose deflations are epimorphisms, then $\mathscr{E}$ endowed with the class of $\mathbb{E}$-conflations becomes an exact category.

Definition 4.1.19. Let $\mathscr{D} \subseteq \mathscr{C}$ be a full additive subcategory, closed under isomorphisms. The subcategory $\mathscr{D}$ is said to be *extension-closed* if, for any conflation $A \rightarrowtail B \twoheadrightarrow C$ which satisfies $A, C \in \mathscr{D}$, then $B \in \mathscr{D}$.

Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category, and let $\mathscr{D} \subseteq \mathscr{C}$ be an extension-closed subcategory.

Proposition 4.1.20. *Let $\mathbb{E}_{\mathscr{D}}$ to be the restriction of $\mathbb{E}$ to $\mathscr{D}^{\mathrm{op}} \times \mathscr{D}$, and define $\mathfrak{s}_{\mathscr{D}}$ by restricting $\mathfrak{s}$, then $(\mathscr{D}, \mathbb{E}_{\mathscr{D}}, \mathfrak{s}_{\mathscr{D}})$ is extriangulated.*

4.2 Auslander–Reiten theory.

In the article [INP18], in collaboration with Osamu Iyama and Hiroyuki Nakaoka, we investigate the study of Auslander–Reiten theory (following [AR75, AR74]) in extriangulated categories, initiated by Panyue Zhou and Bin Zhu in [ZZ18, Section 4].

Our main aims were twofold. First, to show that several classical results concerning existence of almost-split sequences also hold in extriangulated categories. Second, to show that existence of almost-split sequences is inherited under various constructions (relative extriangulated structures, ideal quotients and extension-closed subcategories), making it easier to prove that a given extriangulated category has almost-split sequences.

4.2.1 Almost-split extensions and almost-split sequences.

Definition 4.2.1. An *almost-split* $\mathbb{E}$-extension $\delta \in \mathbb{E}(Z,X)$ is a non-split $\mathbb{E}$-extension such that:

(AS1) $f_*\delta = 0$ for any non-section $f \in \mathscr{C}(X,X')$.

(AS2) $g^*\delta = 0$ for any non-retraction $g \in \mathscr{C}(Z',Z)$.

An *almost-split sequence* is a conflation $X \overset{x}{\rightarrowtail} Y \overset{y}{\twoheadrightarrow} Z$ realizing an almost-split extension $\delta \in \mathbb{E}(Z,X)$.

Definition 4.2.2. A non-zero object $X \in \mathscr{C}$ is said to be *endo-local* if $\mathscr{C}(X,X)$ is local.

Proposition 4.2.3. *For any non-split extriangle $X \overset{x}{\rightarrowtail} Y \overset{y}{\twoheadrightarrow} Z \overset{\delta}{\dashrightarrow}$, the following holds.*

(1) *If δ satisfies* (AS1)*, then X is endo-local.*

(2) *The extension δ satisfies* (AS1) *if and only if x is left almost-split.*

(3) *If δ satisfies* (AS1)*, then y is right minimal.*

Dual statements concerning condition (AS2) *hold.*

Definition 4.2.4. An object $P \in \mathscr{C}$ is called *projective* if, for any $X \in \mathscr{C}$, we have $\mathbb{E}(P,X) = 0$. Dually, an object $I \in \mathscr{C}$ is called *injective* if for any $X \in \mathscr{C}$, we have $\mathbb{E}(X,I) = 0$.

Definition 4.2.5. The extriangulated category $\mathscr{C}$ is said to *have right almost split extensions* if for any endo-local, non-projective object $Z \in \mathscr{C}$, there exists an almost split extension $\delta \in \mathbb{E}(Z,X)$ for some $X \in \mathscr{C}$. Dually, we say that $\mathscr{C}$ *has left almost split extensions* if for any endo-local, non-injective object $X \in \mathscr{C}$, there exists an almost split extension $\delta \in \mathbb{E}(Z,X)$ for some $Z \in \mathscr{C}$. We say that $\mathscr{C}$ *has almost split extensions* if it has right and left almost split extensions.

4.2.2 Auslander–Reiten–Serre duality.

One of the important features of Auslander–Reiten theory is the so-called Auslander–Reiten formula, relating first extensions in module categories to morphism spaces in (co-)stable categories. We show that Auslander–Reiten formula holds for Krull–Schmidt extriangulated categories. Because extriangulated categories generally do not have enough projectives or injectives, one might think of two competing definitions for the (co-)stable category. The most relevant one seems to be:

Definition 4.2.6. We denote by $\mathcal{P}$ (respectively, $\mathcal{I}$) the ideal of $\mathscr{C}$ consisting of all morphisms f satisfying $\mathbb{E}(f,-)=0$ (respectively, $\mathbb{E}(-,f)=0$). The *stable category* (respectively, *costable category*) of $\mathscr{C}$ is defined as the ideal quotient

$$\underline{\mathscr{C}} := \mathscr{C}/\mathcal{P} \text{ (respectively, } \overline{\mathscr{C}} := \mathscr{C}/\mathcal{I}).$$

Remark 4.2.7. (a) The bifunctor $\mathbb{E}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab$ induces a bifunctor $\mathbb{E}\colon \underline{\mathscr{C}}^{\mathrm{op}} \times \overline{\mathscr{C}} \to Ab$.

(b) An object $P \in \mathscr{C}$ is projective if and only if $\underline{\mathscr{C}}(P,-)=0$, if and only if $P \cong 0$ in $\underline{\mathscr{C}}$.

(c) An object $I \in \mathscr{C}$ is injective if and only if $\overline{\mathscr{C}}(-,I)=0$, if and only if $I \cong 0$ in $\overline{\mathscr{C}}$.

A key lemma in several of our proofs is the following:

Lemma 4.2.8. *Let $0 \neq \delta \in \mathbb{E}(Z,X)$ be any $\mathbb{E}$-extension satisfying* (AS1). *Then the following holds for any $A \in \mathscr{C}$.*

(a) *For any $0 \neq \alpha \in \mathbb{E}(A,X)$, there exists $c \in \mathscr{C}(Z,A)$ such that $\delta = c^*\alpha$.*

(b) *For any $0 \neq \overline{a} \in \overline{\mathscr{C}}(A,X)$, there exists $\gamma \in \mathbb{E}(Z,A)$ such that $\delta = a_*\gamma$.*

Definition 4.2.9. Let $(\mathscr{C},\mathbb{E},\mathfrak{s})$ be a $\mathbb{K}$-linear extriangulated category.

(a) A *right Auslander–Reiten–Serre duality* is a pair (τ,η) of an additive functor $\tau\colon \underline{\mathscr{C}} \to \overline{\mathscr{C}}$ and a binatural isomorphism

$$\eta_{X,Y}\colon \underline{\mathscr{C}}(X,Y) \simeq \mathbb{D}\mathbb{E}(Y,\tau X) \quad \text{for any} \quad X,Y \in \mathscr{C}.$$

(b) If moreover τ is an equivalence, we say that (τ,η) is an *Auslander–Reiten–Serre duality*.

Theorem 4.2.10. *Let $\mathscr{C}$ be a $\mathbb{K}$-linear, Ext-finite, Krull-Schmidt, extriangulated category. Then the following conditions are equivalent.*

1. *$\mathscr{C}$ has almost split extensions.*

2. *$\mathscr{C}$ has an Auslander–Reiten–Serre duality.*

4.2.3 Stable module theory.

In this section, we show that the existence of almost-split sequences can be characterised in terms of the structure of the stable and the costable category. For this, we do not need to assume that $\mathscr{C}$ is Krull–Schmidt.

We first note that, since $\mathscr{C}$ typically does not have weak kernel (e.g. when $\mathscr{C}$ is an extension-closed subcategory of a triangulated category) the category of finitely presented modules (coherent functors) over $\mathscr{C}$ is not closed under kernels, hence not abelian. However, the situation becomes much more friendly when passing to the stable category:

Theorem 4.2.11. *Let $\mathscr{C}$ be an extriangulated category with enough projectives and enough injectives. Then the category $\operatorname{mod}\underline{\mathscr{C}}$ is an abelian category with enough projectives and enough injectives,*

$$\operatorname{proj}\underline{\mathscr{C}} = \operatorname{add}\{\underline{\mathscr{C}}(-, A) \mid A \in \mathscr{C}\}, \qquad \operatorname{inj}\underline{\mathscr{C}} = \operatorname{add}\{\mathbb{E}(-, A) \mid A \in \mathscr{C}\}.$$

Moreover, we have equivalences $\underline{\mathscr{C}} \to \operatorname{proj}\underline{\mathscr{C}}$ given by $A \mapsto \underline{\mathscr{C}}(-, A)$ and $\overline{\mathscr{C}} \to \operatorname{inj}\underline{\mathscr{C}}$ given by $A \mapsto \mathbb{E}(-, A)$ up to direct summands.

The following consequence is worth noting:

Proposition 4.2.12. *Let $\mathscr{C}$ be a $\mathbb{K}$-linear extriangulated category with enough projectives and injectives. Then $\mathscr{C}$ is Ext-finite if and only if $\overline{\mathscr{C}}$ is Hom-finite if and only if $\underline{\mathscr{C}}$ is Hom-finite.*

Let $\mathscr{D}$ be a $\mathbb{K}$-linear additive category. Then any $\mathscr{D}$-module F can be regarded as a functor $F \colon \mathscr{D} \to \operatorname{mod}\mathbb{K}$. We define a $\mathscr{D}^{\mathrm{op}}$-module $\mathbb{D}F$ as the composition $\mathscr{D} \xrightarrow{F} \operatorname{mod}\mathbb{K} \xrightarrow{\mathbb{D}} \operatorname{mod}\mathbb{K}$.

Definition 4.2.13. [AR74] We call $\mathscr{D}$ a *dualizing $\mathbb{K}$-variety* if the following conditions hold.

(a) The category $\mathscr{D}$ is Hom-finite over $\mathbb{K}$ and idempotent complete.

(b) For any $F \in \operatorname{mod}\mathscr{D}$, we have $\mathbb{D}F \in \operatorname{mod}\mathscr{D}^{\mathrm{op}}$.

(c) For any $G \in \operatorname{mod}\mathscr{D}^{\mathrm{op}}$, we have $\mathbb{D}G \in \operatorname{mod}\mathscr{D}$.

In this case, we have an equivalence $\mathbb{D}\colon \operatorname{mod}\mathscr{D} \simeq \operatorname{mod}\mathscr{D}^{\mathrm{op}}$.

Theorem 4.2.14. *Let $\mathscr{C}$ be a $\mathbb{K}$-linear, Ext-finite extriangulated category with enough projectives and enough injectives such that $\overline{\mathscr{C}}$ and $\underline{\mathscr{C}}$ are idempotent complete. Then the following conditions are equivalent.*

1. *The extriangulated category $\mathscr{C}$ has an Auslander–Reiten–Serre duality.*
2. *The $\mathbb{K}$-linear category $\overline{\mathscr{C}}$ is a dualizing $\mathbb{K}$-variety.*

3. *The $\mathbb{K}$-linear category $\underline{\mathscr{C}}$ is a dualizing $\mathbb{K}$-variety.*

Remark 4.2.15. In [INP18, Section 7], we further investigate the structure of the stable category, when $\mathscr{C}$ is Krull-Schmidt and has left almost-split sequences. We show that $\underline{\mathscr{C}}$ has the structure of a τ-category [Iya05], which implies the Radical Layers Theorem [IT84, Iya05].

4.2.4 Stability of the existence of almost-split sequences.

In order to prove that a given extriangulated category has almost-split sequence, the following strategy is often useful: Show that the extriangulated category is obtained from an exact or triangulated category with almost-split sequences by performing various categorical constructions that preserve existence of almost-split sequences. We consider here: passage to relative extriangulated structures, to ideal quotients and to extension-closed subcategories.

Definition 4.2.16. Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. An additive subfunctor $\mathbb{F}$ of $\mathbb{E}$ is called a *closed subfunctor* if $\mathbb{F}$-inflations are closed under composition or equivalently if $\mathbb{F}$-deflations are closed under composition.

The interest in that property comes from:

Proposition 4.2.17. [HLN17, Proposition 3.14] *Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category, and let $\mathbb{F}$ be an additive subfunctor of $\mathbb{E}$. Then the restriction $(\mathscr{C}, \mathbb{F}, \mathfrak{s}|_{\mathbb{F}})$ is extriangulated if and only if the subfunctor $\mathbb{F}$ is closed.*

Remark 4.2.18. Without assuming the additive subfunctor to be closed, $(\mathscr{C}, \mathbb{F}, \mathfrak{s}|_{\mathbb{F}})$ is "pre-extriangulated" in the sense that all axioms but (ET4) and (ET4$^{\mathrm{op}}$) hold.

Proposition 4.2.19. *Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category having almost-split extensions, and let $\mathbb{F}$ be an additive subfunctor of $\mathbb{E}$. Then $(\mathscr{C}, \mathbb{F}, \mathfrak{s}|_{\mathbb{F}})$ has almost-split extensions.*

Remark 4.2.20. Surprisingly, this proposition is used in [PPPP19, Section 3] in order to prove that the type cone of $\boldsymbol{g}$-vector fans of cluster algebras of finite type is simplicial. Indeed, in that proof, we are led to consider the relative extriangulated structure of cluster categories given by extensions that factor through some given cluster-tilting object.

Proposition 4.2.21. *Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and let $\mathcal{D} \subseteq \mathscr{C}$ be a strictly full, additive subcategory such that all objects in $\mathcal{D}$ are both projective and injective. Let $\delta \in \mathbb{E}(Z, X)$ be an almost-split extension. Then δ induces an almost-split extension in $\mathscr{C}/\mathcal{D}$ (endowed with its extriangulated structure canonically induced from that of $\mathscr{C}$).*

Our last result is inspired from [Jør09].

Proposition 4.2.22. *Let $\mathscr{C}$ be a Krull–Schmidt extriangulated category, and let $\mathcal{D} \subseteq \mathscr{C}$ be a contravariantly finite, full, additive, extension-closed subcategory. If $\mathscr{C}$ has almost-split sequences, then so has $\mathcal{D}$.*

The following consequence is used in [PPPP19, Section 4], in relation with gentle algebras.

Proposition 4.2.23. *Let Λ be an Artin algebra and let $K^{[-1,0]}(\operatorname{proj}\Lambda)$ be the full subcategory of the homotopy category $K^b(\operatorname{proj}\Lambda)$ consisting of complexes concentrated in degrees -1 and 0 (using cohomological conventions). Then $K^{[-1,0]}(\operatorname{proj}\Lambda)$ has almost-split sequences.*

4.3 Applications to gentle algebras.

In this section, we discuss two results that illustrate the application of the theory of extriangulated categories to the study of gentle algebras. The first one is taken from [PPPP19, Section 4] and describes an abstract setting in which holds an analogue of a result of M. Auslander describing minimal relations for the Grothendieck groups of module categories over Artin algebras [Aus84]. The second result is part of a work in progress with Osamu Iyama, Hiroyuki Nakaoka and Salvatore Stella, and describes an extriangulated categorification of all bending walks (see Section 3.1) on a gentle algebra.

4.3.1 Relations for Grothendieck groups.

Assumption 4.3.1. Fix an extriangulated category $\mathscr{C}$ with a strictly full, additive subcategory $\mathcal{T}$, stable under taking direct summands, and satisfying the following three properties:

1. Every $T \in \mathcal{T}$ is projective in $\mathscr{C}$;
2. For each $T \in \mathcal{T}$, the morphism $T \to 0$ is an inflation;
3. For each $X \in \mathscr{C}$, there is an extriangle $T_1^X \rightarrowtail T_0^X \twoheadrightarrow X \overset{\delta_X}{\dashrightarrow}$ in $\mathscr{C}$ with T_0^X, T_1^X in $\mathcal{T}$.

Example 4.3.2. Examples of categories satisfying Assumption 4.3.1 are:

- the 2-Calabi–Yau triangulated categories admitting a cluster-tilting subcategory $\mathcal{T}$, endowed with the relative extriangulated structure given by extensions factoring through $\Sigma\mathcal{T}$;
- for any Artin algebra Λ, the category $K^{[-1,0]}(\operatorname{proj}\Lambda)$ of complexes of finitely generated projective Λ-modules concentrated in degrees -1 and 0, with morphisms considered up to homotopy;

Remark 4.3.3. For any object $T \in \mathcal{T}$, we fix an extriangle $T \rightarrowtail 0 \twoheadrightarrow \Sigma T \dashrightarrow$. This notation extends to an equivalence of categories from the category $\mathcal{T}$ of projective objects in $\mathscr{C}$ to the category $\Sigma\mathcal{T}$ of injective objects in $\mathscr{C}$.

Notation 4.3.4. We let $F : \mathscr{C} \to \operatorname{Mod}\mathcal{T}$ be the functor defined on objects by sending $X \in \mathscr{C}$ to $\mathscr{C}(-, X)|_{\mathcal{T}}$.

Lemma 4.3.5. *For any $X \in \mathscr{C}$, the functor FX is finitely presented. We thus have a functor*

$$F : \mathscr{C} \to \operatorname{mod} \mathcal{T}.$$

Proposition 4.3.6 below extends similar results from [BMR07, KR07, KZ08, IY08] to the setting under consideration.

Proposition 4.3.6. *The functor F induces an equivalence of categories*

$$F : \mathscr{C}/(\Sigma\mathcal{T}) \to \operatorname{mod} \mathcal{T}$$

where $(\Sigma\mathcal{T})$ is the ideal of morphisms factoring through an object of the form ΣT, for some $T \in \mathcal{T}$.

Definition 4.3.7. We let $K_0(\mathscr{C})$ denote the *Grothendieck group* of $\mathscr{C}$, that is, the quotient of the free abelian group generated by symbols $[X]$, for each $X \in \mathscr{C}$, by the relations $[X] - [Y] + [Z]$, for each conflation $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathscr{C}$.

Remark 4.3.8. Since $\mathcal{T}$ is extension-closed in $\mathscr{C}$, it inherits an extriangulated structure. Because $\mathcal{T}$ is made of projective objects in $\mathscr{C}$, its extriangulated structure splits and we have $K_0(\mathcal{T}) \cong K_0^{\mathrm{sp}}(\mathcal{T})$.

The notion of index from [DK08, Pal08] generalises to our current setting.

Definition 4.3.9. For any object $X \in \mathscr{C}$, fix some extriangle $T_1^X \rightarrowtail T_0^X \twoheadrightarrow X \overset{\delta_X}{\dashrightarrow}$ and define the *index* of X by

$$\operatorname{ind}_{\mathcal{T}} X = [T_0^X] - [T_1^X] \in K_0(\mathcal{T}).$$

Proposition 4.3.10. *The assignment $X \mapsto \operatorname{ind}_{\mathcal{T}} X$ is well-defined and induces an isomorphism of abelian groups:*

$$\operatorname{ind}_{\mathcal{T}} : K_0(\mathscr{C}) \xrightarrow{\cong} K_0(\mathcal{T}).$$

We assume moreover that $\mathscr{C}$ is Krull–Schmidt, $\mathbb{K}$-linear, Ext-finite, and has Auslander–Reiten–Serre duality (Section 4.2), and that the subcategory $\mathcal{T}$ is of the form $\operatorname{add} T$, where $T = T_1 \oplus \cdots \oplus T_n$ is a basic object.

Notation 4.3.11. For any $X, Y \in \mathscr{C}$, let

$$\langle X, Y\rangle := \dim_{\mathbb{K}} \operatorname{Hom}_{\mathcal{T}}(FX, FY) = \dim_{\mathbb{K}} \mathscr{C}/(\Sigma\mathcal{T})(X, Y).$$

For any almost-split sequence $X \rightarrowtail E \twoheadrightarrow Y \dashrightarrow$, define

$$\ell_X := [X] + [Y] - [E] \in K_0^{\mathrm{sp}}(\mathscr{C}).$$

Theorem 4.3.12. *Assume that $\mathscr{C}$ is a $\mathbb{K}$-linear, Ext-finite, Krull–Schmidt, extriangulated category with Auslander–Reiten–Serre duality. Assume that T is a projective object of $\mathscr{C}$ such that any $X \in \mathscr{C}$ admits a conflation $T_1^X \rightarrowtail T_0^X \twoheadrightarrow X$ with $T_0^X, T_1^X \in \operatorname{add}(T)$, and the morphism $T \to 0$ is an inflation. Fix a conflation $T \to 0 \to \Sigma T$. Then $\mathscr{C}$ has only finitely many isomorphism classes of indecomposable objects if and only if the set*

$$L := \{\ell_X \mid X \in \operatorname{ind}(\mathscr{C}) \setminus \operatorname{add}(\Sigma T)\}$$

generates the kernel of the canonical projection $\boldsymbol{g} : K_0^{\mathrm{sp}}(\mathscr{C}) \to K_0(\mathscr{C})$. In this case, the set L is a basis of the kernel of $\boldsymbol{g}$, and for any $x \in \ker(\boldsymbol{g})$, we have that

$$x = \sum_{A \in \operatorname{ind}(\mathscr{C}) \setminus \operatorname{add}(\Sigma T)} \frac{\langle x, A \rangle}{\langle \ell_A, A \rangle} \ell_A.$$

Our main motivation for Theorem 4.3.12 was Corollary 4.3.13 below which, when applied to the specific case of (2-acyclic, brick) gentle algebras, shows that the type cone of their $\boldsymbol{g}$-vector fans is simplicial.

Corollary 4.3.13. *Under the assumptions of Theorem 4.3.12, suppose moreover that* $\operatorname{ind}(\mathscr{C})$ *is finite. Let $X \rightarrowtail E \twoheadrightarrow Y \dashrightarrow$ be any extriangle. Then the element $x = [X] + [Y] - [E]$ of the kernel of $\boldsymbol{g}$ is a non-negative linear combination of the ℓ_A, with $A \in \operatorname{ind}(\mathscr{C}) \setminus \operatorname{add}(\Sigma T)$.*

4.3.2 Simplicial type cones from gentle algebras.

In this section, we explain, following [PPPP19, Section 4], how to apply Theorem 4.3.12 and Corollary 4.3.13 to study the type cone of the $\boldsymbol{g}$-vector fans of gentle algebras (see Section 2.3.1).

Lemma 4.3.14. *Let Λ be any finite-dimensional $\mathbb{K}$-algebra. Then the category $K^{[-1,0]}(\operatorname{proj}\Lambda)$ is an extriangulated category satisfying all assumptions of Theorem 4.3.12.*

Definition 4.3.15. Recall that a two-term silting object is a complex T in $K^{[-1,0]}(\operatorname{proj}\Lambda)$ such that $\operatorname{Hom}_{K^b}(T, \Sigma T) = 0$ and the number of isomorphism classes of indecomposable summands of T is the same as that of Λ.

Remark 4.3.16. The definition given above is only equivalent to the usual definition of a silting object for two-term complexes of projectives.

Definition 4.3.17. A conflation $X \rightarrowtail E \twoheadrightarrow Y$ of $K^{[-1,0]}(\operatorname{proj}\Lambda)$ is called a *mutation conflation* if there are basic, two-term, silting objects $X \oplus R$, $Y \oplus R$, with X and Y indecomposable, such that the inflation $X \rightarrowtail E$ is a left $(\operatorname{add} R)$-approximation.

Remark 4.3.18. In Definition 4.3.17, the requirement that X and Y are indecomposable implies the map $E \twoheadrightarrow Y$ is a right $(\operatorname{add} R)$-approximation, and that both approximations are minimal.

Proposition 4.3.19. *Let Λ be a finite-dimensional $\mathbb{K}$-algebra, where $\mathbb{K}$ is a field. Let X and Y be objects of $K^{[-1,0]}(\operatorname{proj}\Lambda)$.*

1. *If there is a mutation conflation (see Definition 4.3.17) of the form $X \rightarrowtail E \twoheadrightarrow Y$, then there can be no mutation conflation of the form $Y \rightarrowtail E' \twoheadrightarrow X$.*
2. *Assume that $X \rightarrowtail E \twoheadrightarrow Y$ and $X \rightarrowtail E' \twoheadrightarrow Y$ are two mutation conflations. Then E and E' are isomorphic.*

Corollary 4.3.20. *The $\boldsymbol{g}$-vector fans of gentle algebras satisfy the unique exchange relation property (see Definition 2.3.10).*

Theorem 4.3.21. *Let Λ be an Artin algebra all of whose indecomposable objects are τ-rigid bricks. Then the almost-split conflations of $K^{[-1,0]}(\operatorname{proj}\Lambda)$ are mutation conflations if and only if for any non-projective indecomposable Λ-module M, the space $\operatorname{Hom}_\Lambda(M, \tau^2 M)$ vanishes.*

Corollary 4.3.22. *Let Λ be a finite dimensional brick algebra of finite representation type. Assume moreover that, for any indecomposable Λ-module M, the space $\operatorname{Hom}_\Lambda(M, \tau^2 M)$ vanishes. Then the support τ-tilting fan of Λ has the unique exchange relation property (see Definition 2.3.10) and its type cone is simplicial (see Section 2.3.1).*

As a consequence of Corollary 4.3.22, we give an explicit description of all realizations of the support τ-tilting fan of Λ. Another consequence is an algebraic proof of Corollary 4.3.24 below, that we also proved by purely combinatorial methods.

Definition 4.3.23. A gentle algebra is called *brick* if all its indecomposable representations are bricks, and *2-acyclic* if its Gabriel quiver does not contain any oriented 2-cycle.

Corollary 4.3.24. *For any brick and 2-acyclic gentle algebra, the type cone of its support τ-tilting fan is simplicial.*

4.3.3 The extriangulated category of walks.

This section, presenting a work in progress with Hiroyuki Nakaoka, is motivated by the following remark. Let G be a grid, and let (Q, I) be the associated gentle algebra. Then, there is a bijection between non-terminal, bending walks in G and isoclasses of indecomposable representations of (Q, I). It is thus quite tempting to draw the Auslander–Reiten quiver of (Q, I), but to replace the indecomposable at each vertex by the corresponding walk. Once this is done, it becomes rather obvious that irreducible morphisms between indecomposable representations correspond to some elementary moves on the walks, similarly to [CCS06] for instance. A formal proof is given in [PPP17, Proposition 2.49].

At a first glance, this has very little interest, because such a combinatorial description of the Auslander–Reiten quiver, by using elementary moves on strings, was already known. However, I would like to point out two interesting facts. First, the elementary moves for strings might be of two different sorts (add a hook, or remove a cohook), making the combinatorics slightly less agreable, whereas the elementary moves for walks are completely uniform. Second, it is now possible to expand the Auslander–Reiten quiver by including straight walks, and terminal walks: See Figure 4.1.

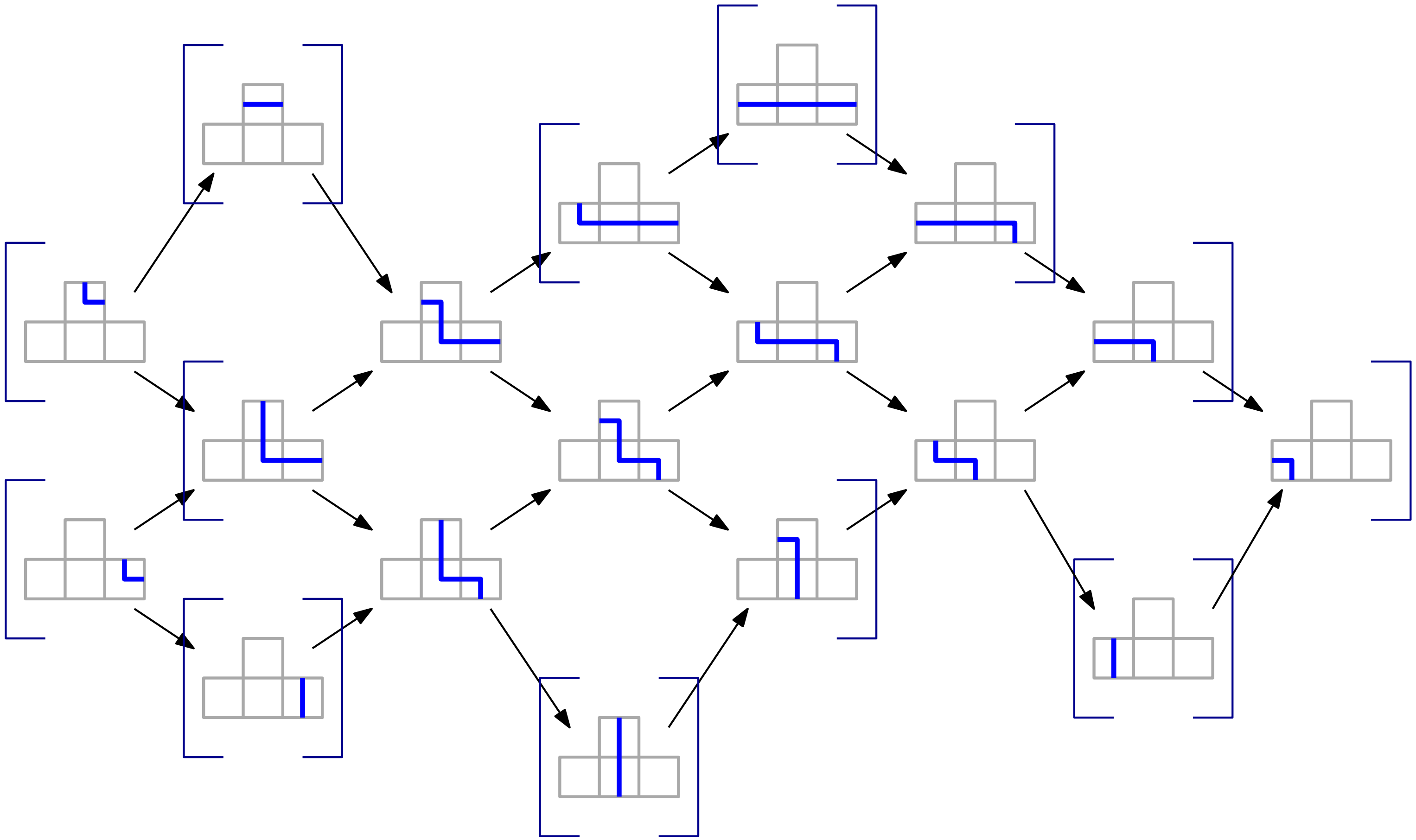


Figure 4.1: The combinatorial translation quiver associated with a grid. Each row corresponds to an orbit for the translation; it starts at an injective vertex (indicated by a right bracket) and ends at a projective vertex (left bracket).

Question 4.3.25. Is it possible to prove that this translation quiver is the Auslander–Reiten quiver of some Krull–Schmidt, exact category?

We would like the straight walks to coincide with the isoclasses of indecomposable projective-injective objects; initial walks to coincide with the isoclasses of indecomposable projective non-injective objects; and the terminal walks to coincide with the isoclasses of indecomposable injective non-projective objects. Note that we are mainly interested in the reduced non-kissing complex, i.e. in the non-straight walks. It thus makes more sense to study the ideal quotient, obtained by killing the projective-injectives, of the conjectural exact category above. The results of Section 4.2.4 would apply and show that this quotient is an extriangulated category with almost-split sequences.

Let (Q, I) be a gentle bound quiver.

Definition 4.3.26. The exact category of walks is defined as the full subcategory $\mathscr{W}$ of $\operatorname{mod} A^{❀}$ on those objects M that satisfy $\operatorname{Hom}_{A^{❀}}(\tau^{-1} \operatorname{soc} P, M) = 0$ for any $P \in \operatorname{proj} S^{❀}$ and $\operatorname{Hom}_{A^{❀}}(Q, \tau M) = 0$ for any projective-injective $A^{❀}$-module Q.

Remark 4.3.27. The subcategory $\mathscr{W}$ is extension-closed in $\operatorname{mod} A^{❀}$, and is thus an exact category.

The next lemma explains the name for $\mathscr{W}$.

Lemma 4.3.28. *Let σ be a string for the gentle bound quiver $(Q^{❀}, I^{❀})$. Then the string module M_σ belongs to $\mathscr{W}$ if and only if σ is a walk for (Q, I).*

The next two results illustrate the fact that $\mathscr{W}$ categorifies the combinatorics of walks.

Lemma 4.3.29. *Let ω be a maximal string for $(Q^{❀}, I^{❀})$. The following properties holds:*

- *The projective objects of $\mathscr{W}$ are precisely the non-simple projective $A^{❀}$-modules.*
- *The injective objects of $\mathscr{W}$ are precisely the non-simple injective $A^{❀}$-modules.*
- *The string module M_ω is projective and non-injective in $\mathscr{W}$ if and only if ω is a projective (initial) walk.*
- *The string module M_ω is injective and non-projective in $\mathscr{W}$ if and only if ω is a shifted projective (terminal) walk.*
- *The string module M_ω is projective and injective in $\mathscr{W}$ if and only if ω is a straight walk.*

Proposition 4.3.30. *Let $\mathscr{W}^{\mathrm{red}}$ be the ideal quotient of $\mathscr{W}$ by the projective-injectives, and let ω, ω' be two walks on (Q, I). Then $\mathbb{E}(M_{\omega'}, M_\omega) \neq 0$ in the extriangulated category $\mathscr{W}^{\mathrm{red}}$ if and only if the walk ω kisses the walk ω'.*

Chapter 5

Homotopical algebra.

Model categories were introduced by Daniel G. Quillen [Qui67] as an axiomatization of homotopy theory and of homological algebra. Model category structures can be thought of as an enrichement of triangulated categories, which is particularly well-adapted to the computation of homotopy limits and colimits. Homotopical algebra is also the language that is used in order to compare different models for infinity-categories. In this chapter, we present two results relating to model category structures. In the first section, we shed new lights on results by Aslak B. Buan and Robert J. Marsh on categories of representations of endo-rigid algebras [BM13, BM12]. In the second section, we extend Hovey's correspondence [Hov07, Gil11] to the case of extriangulated categories. Our main result is that the homotopy category of an exact model structure is always triangulated.

5.1 From triangulated categories to module categories via Homotopical algebra.

Our aim in this section is to give a homotopical algebraic interpretation of a result of Aslak Buan and Robert Marsh [BM13] (see also [BM12, Bel13]) on some localisations, associated with rigid objects, of triangulated categories. This inspired the homotopical algebraic part [JM17] of Lucie Jacquet-Malo's PhD thesis, which deals with the surprisingly more technical case of exact categories; a setting in which the results of Aslak Buan and Robert Marsh were not previously known.

5.1.1 Endomorphism algebras of rigid objects.

The interest in rigid objects of module categories and of triangulated categories arose in tilting theory. The study of cluster algebras revived this interest: Rigid objects in cluster categories categorify the cluster monomials of an associated cluster algebra.

Let $\mathscr{C}$ be a triangulated category with suspension functor Σ. An object T in $\mathscr{C}$ is called *rigid* if it has no non-trivial self-extensions, i.e. if $\mathscr{C}(T, \Sigma T) = 0$. A rigid object T is called *maximal rigid* if moreover $T \oplus X$ is rigid if and only if $X \in \operatorname{add} T$, where $\operatorname{add} T$ is the full subcategory of $\mathscr{C}$ whose objects are the direct summands of finite directs sums of copies of T. One nice feature of rigid objects in a triangulated category is that all the information concerning their representation theory is contained in the triangulated category:

Theorem 5.1.1 (Buan–Marsh–Reiten [BMR07]). *Let $\mathbb{K}$ be a field, let Q be an acyclic quiver, and let $\mathscr{C} = D^b(\mathbb{K}Q)/\tau^{-1}[1]$ be the associated cluster category [BMR+06]. If T is a maximal rigid object of $\mathscr{C}$, then the functor $\mathscr{C}(T,-)$ induces an equivalence of categories:*

$$\mathscr{C}/(\Sigma T) \xrightarrow{\simeq} \operatorname{mod}\operatorname{End}_{\mathscr{C}}(T)^{op},$$

where (ΣT) denotes the ideal of morphisms factoring through $\operatorname{add}\Sigma T$.

We note that, if $X \in \mathscr{C}$, then the $\operatorname{mod}\operatorname{End}_{\mathscr{C}}(T)^{\mathrm{op}}$-module structure on the finite dimensional $\mathbb{K}$-vector space $\mathscr{C}(T,X)$ is given by precomposition. Theorem 5.1.1 was generalised in various directions; see for instance [KZ08, KR07, IY08, JJ19]. In particular, it is shown in [KZ08] that the abelian structure on $\mathscr{C}_Q/(\Sigma T)$ can be described in terms of the triangulated structure [Kel05] of $\mathscr{C}_Q$.

Let us state a generalisation due to Osamu Iyama and Yuji Yoshino [IY08], which was the starting point for the work of Aslak Buan and Robert Marsh in [BM13].

Theorem 5.1.2 (Iyama–Yoshino [IY08]). *Let $\mathbb{K}$ be a field and let $\mathscr{C}$ be a $\mathbb{K}$-linear, Hom-finite, Krull–Schmidt, triangulated category with some rigid object T. The full subcategory of $\mathscr{C}$ whose objects are cones of morphisms in* $\operatorname{add} T$ *is denoted by $T*\Sigma T$. Then the functor $\mathscr{C}(T,-)$ induces an equivalence of categories:*

$$T * \Sigma T/(\Sigma T) \xrightarrow{\simeq} \operatorname{mod}\operatorname{End}_{\mathscr{C}}(T)^{op}.$$

From this result arise the following questions: What are the properties of the functor $\mathscr{C}(T,-) : \mathscr{C} \to \operatorname{mod}\operatorname{End}_{\mathscr{C}}(T)^{\mathrm{op}}$? Is it possible to describe the module category $\operatorname{mod}\operatorname{End}_{\mathscr{C}}(T)^{\mathrm{op}}$ from $\mathscr{C}$, without computing the subcategory $T * \Sigma T$?

The answer given by Aslak Buan and Robert Marsh is that $\mathscr{C}(T,-)$ is a localisation functor.

5.1.2 Localisations.

The following situation arises in various fields of mathematics. Assume that $\mathscr{C}$ is a category with some class $\mathcal{W}$ of morphisms, called weak equivalences. If there is a weak equivalence from X to Y, one would like to think of X and Y as being isomorphic. For example, $\mathscr{C}$ might be the category of complexes of modules over some ring, and $\mathcal{W}$ the class of quasi-isomorphisms (morphisms inducing isomorphisms on homologies). Or $\mathscr{C}$ might be the category of compactly generated (weak Hausdorff) topological spaces, with $\mathcal{W}$ the class of weak (homotopy) equivalences: the morphisms inducing bijections on homotopy groups, for all choices of a base point.

There is a method [GZ67] for constructing a new category $\mathscr{C}[\mathcal{W}^{-1}]$ having the same objects as $\mathscr{C}$ but where morphisms in $\mathcal{W}$ become isomorphisms.

Definition 5.1.3. A *localisation* of $\mathscr{C}$ at $\mathcal{W}$ is the datum of a functor $\mathscr{C} \xrightarrow{L} \mathscr{C}[\mathcal{W}^{-1}]$ with the property that, for any functor $\mathscr{C} \xrightarrow{F} \mathcal{D}$ such that Fw is an isomorphism in $\mathcal{D}$ whenever w is in $\mathcal{W}$, there is a unique functor $\mathscr{C}[\mathcal{W}^{-1}] \xrightarrow{G} \mathcal{D}$ such that $GL = F$:

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "L"'] & \mathcal{D} \\ \mathscr{C}[\mathcal{W}^{-1}] \arrow[ur, dashed, "G"'] & \end{tikzcd}$$

We note that the diagram above is required to commute "on the nose" and not only up to some natural isomorphism. In particular, the category $\mathscr{C}[\mathcal{W}^{-1}]$, if it exists, is unique up to isomorphism (and not just up to equivalence). However, this is mostly a matter of taste and an "up to equivalence" version of the definition also exists in the literature.

The recipe given in [GZ67] for constructing $\mathscr{C}[\mathcal{W}^{-1}]$ can be sketched as follows: Consider all words on (composable) morphisms of $\mathscr{C}$ and formal inverses w^{-1} to all morphisms w in $\mathcal{W}$, up to the equivalence relation obtained by identifying subwords of the form fg and $f \circ g$, $1f$ or $f1$ and f, and ww^{-1} or $w^{-1}w$ and 1. The "category" with objects the objects of $\mathscr{C}$, with morphisms the equivalence classes of words, and with composition induced by concatenation of words is a localisation of $\mathscr{C}$ at $\mathcal{W}$. Unfortunately, there is some set-theoretic issue with this construction: The collection of all morphisms between two objects might form a proper class rather than a set. As shown by Theorem 5.1.4, this issue does not arise in the setup considered in [BM13].

Let $\mathscr{C}$ be, as in Theorem 5.1.2, a $\mathbb{K}$-linear, Hom-finite, Krull–Schmidt, triangulated category, and let $T \in \mathscr{C}$ be rigid. We write $T^{\perp}$ for the full subcategory of $\mathscr{C}$ whose objects X satisfy $\mathscr{C}(T, X) = 0$. Let $\mathcal{S}$ be the class of morphisms $X \xrightarrow{f} Y$ such that, for some (equivalently, any) triangle $Z \xrightarrow{g} X \xrightarrow{f} Y \xrightarrow{h} \Sigma Z$, both morphisms g and h belong to the ideal $(T^{\perp})$ of morphisms factoring through $T^{\perp}$.

Theorem 5.1.4 (Buan–Marsh [BM13])**.** *Let $\mathscr{C}$ be a $\mathbb{K}$-linear, Hom-finite, Krull–Schmidt, triangulated category with a Serre functor, and let $T \in \mathscr{C}$ be rigid.*

1. *For any morphism $s \in \mathcal{S}$, $\mathscr{C}(T, s)$ is an isomorphism in* $\operatorname{mod}\operatorname{End}_{\mathscr{C}}(T)^{op}$.
2. *The functor $\mathscr{C}[\mathcal{S}^{-1}] \xrightarrow{G} \operatorname{mod}\operatorname{End}_{\mathscr{C}}(T)^{op}$ induced from $\mathscr{C}(T, -)$ is an equivalence of categories.*

In particular, the localisation of $\mathscr{C}$ at $\mathcal{S}$ exists: The construction of [GZ67] is a (small) category. A key lemma in the proof of Theorem 5.1.4 is:

Lemma 5.1.5 (Buan–Marsh [BM13])**.** *Let $X \in \mathscr{C}$. Then there is a triangle $Y \xrightarrow{g} A \xrightarrow{f} X \xrightarrow{h} \Sigma Y$, with $A \in T * \Sigma T$, $Y \in T^{\perp}$ and $h \in (T^{\perp})$. In particular, the modules $\mathscr{C}(T, A)$ and $\mathscr{C}(T, X)$ are isomorphic.*

Under the assumptions of Theorem 5.1.4, we thus obtain two equivalent categories. The first one is the localisation of $\mathscr{C}$ at some class of morphisms $\mathcal{S}$. The second one is the full subcategory $T * \Sigma T$ of $\mathscr{C}$ where morphisms are considered up to some equivalence relations (two morphisms f and g are equivalent if $f - g$ factors through $\operatorname{add}\Sigma T$). This is reminiscent to the theory of model categories [Qui67]. Our aim is to make this analogy more precise: We will give some homotopical algebraic interpretation of Theorem 5.1.4, and of Lemma 5.1.5. Our main motivation for pushing this analogy farther was the hope that it might provide a tool allowing for a generalisation of Theorem 5.1.4 including the case of Hom-infinite cluster categories ([Ami09], [Pla11b]).

5.1.3 Model categories.

Model categories, which axiomatise homotopy theory, were introduced by Daniel G. Quillen in [Qui67]. Let $\mathscr{C}$ be a category and $\mathcal{W}$ a collection of morphisms to be inverted. If $(\mathscr{C}, \mathcal{W})$ can be endowed with a model category structure, then the localisation, called $\operatorname{Ho}\mathscr{C}$, of $\mathscr{C}$ at $\mathcal{W}$ exists (and comes equipped with more structure).

This axiomatic version of homotopy theory was called *homotopical algebra* by Daniel G. Quillen, since it subsumes both homological algebra (when $\mathscr{C}$ is the category of complexes of modules over a ring, $\mathcal{W}$ is the class of quasi-isomorphisms, and $\operatorname{Ho}\mathscr{C}$ is the derived category) and homotopy (e.g. when $\mathscr{C}$ is the category of compactly generated topological spaces, $\mathcal{W}$ is the class of weak equivalences, and $\operatorname{Ho}\mathscr{C}$ is the homotopy category of spaces).

Notation. For two morphisms f, g in a category $\mathscr{C}$, we write $f \,\square\, g$ if, for any commutative square

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & X' \arrow[d, "g"] \\ Y \arrow[r, "b"'] \arrow[ur, dashed, "\alpha"] & Y', \end{tikzcd}$$

there is a lift α such that $\alpha f = a$ and $g\alpha = b$. In that case, we consider f and g to be weakly orthogonal. Thus, if $\mathcal{D}$ is a full subcategory of $\mathscr{C}$, we will use the notations $\mathcal{D}^{\square}$ and ${}^{\square}\mathcal{D}$.

Assume that $\mathscr{C}$ has finite limits and colimits (some authors assume all small limits and colimits). Then a *model category structure* on $\mathscr{C}$ is the datum of three classes $\mathcal{W}, \mathcal{F}ib, \mathcal{C}of$ of morphisms, called respectively weak equivalences, fibrations and cofibrations, satisfying some set of axioms inspired from basic homotopy theory. The first two axioms concern the stability properties of $\mathcal{W}, \mathcal{F}ib, \mathcal{C}of$, and the other two axioms ensure that the three classes interact nicely. More explicitely:

1. The weak equivalences have the *two-out-of-three* property: For any composable f and g, if any two of f, g and gf are weak equivalences, then so is the third.

2. The classes $\mathcal{W}, \mathcal{F}ib$ and $\mathcal{C}of$ contain all identities, are closed under compositions and under taking retracts (in the category of morphisms of $\mathscr{C}$).

3. Lifting properties: $(\mathcal{W} \cap \mathcal{C}of) \,\square\, \mathcal{F}ib$ and $\mathcal{C}of \,\square\, (\mathcal{F}ib \cap \mathcal{W})$.

4. Factorisations: Any morphism belongs both to $\mathcal{F}ib \circ (\mathcal{W} \cap \mathcal{C}of)$ and to $(\mathcal{F}ib \cap \mathcal{W}) \circ \mathcal{C}of$.

By Axiom (4), any morphism f admits two factorisations:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "\sim", "i"'] & & Y \\ & X' \arrow[ur, two heads, "p"'] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "j"'] & & Y \\ & Y' \arrow[ur, two heads, "\sim", "q"'] & \end{tikzcd}$$

where i is a cofibration and a weak equivalence, p is a fibration, j is a cofibration and q is a fibration and a weak equivalence. An object X is *fibrant* if the canonical morphism from X to the terminal object $*$ of $\mathscr{C}$ (which exists since $\mathscr{C}$ has finite limits) is a fibration. Dually, A is *cofibrant* if the canonical morphism from the initial object $\emptyset$ to A is a cofibration. By applying Axiom (4) to $X \to *$ and $\emptyset \to X$, every object X is seen to be weakly equivalent to some fibrant object and to some cofibrant object. These are called respectively *fibrant replacement* and *cofibrant replacement*. Let $\mathscr{C}_{cf}$ be the full subcategory of $\mathscr{C}$ whose objects are both fibrant and cofibrant. In any model category, one can define *path objects*, *cylinder objects* and *homotopies* thus giving an axiomatic version of the corresponding notions for topological spaces. We write $f \simeq_{\mathrm{htp}} g$ if two morphisms f and g are homotopic.

Theorem 5.1.6 (Quillen [Qui67])**.** *Let $\mathscr{C}$ be a model category and let $\mathrm{Ho}\,\mathscr{C}$ be the localisation of $\mathscr{C}$ at the class of weak equivalences. Then:*

(i) For any $X, Y \in \mathscr{C}_{cf}$, homotopy is an equivalence relation on $\mathscr{C}(X, Y)$, compatible with composition.

(ii) The inclusion of $\mathscr{C}_{cf}$ into $\mathscr{C}$ induces an equivalence of categories

$$\mathscr{C}_{cf}/\simeq_{htp} \longrightarrow \mathrm{Ho}\,\mathscr{C}.$$

In particular $\mathrm{Ho}\,\mathscr{C}$ is a (small) category.

There are two similar pictures coming from different setups in Theorems 5.1.4 and 5.1.6:

$$\begin{tikzcd} & \mathscr{C} \ni T \text{ rigid} \arrow[dl, squiggly] \arrow[dr, squiggly] & \\ \frac{T * \Sigma T}{(\Sigma T)} \arrow[rr, "\simeq"] & & \mathscr{C}[\mathcal{S}^{-1}] \end{tikzcd} \qquad \begin{tikzcd} & \mathscr{C} \text{ model category} \arrow[dl, squiggly] \arrow[dr, squiggly] & \\ \mathscr{C}_{cf}/\simeq_{\mathrm{htp}} \arrow[rr, "\simeq"] & & \mathscr{C}[\mathcal{W}^{-1}] \end{tikzcd}$$

Our main result is motivated by this analogy.

5.1.4 Model structures from rigid objects.

Let $\mathscr{C}$ be a triangulated category and let $T \in \mathscr{C}$ be rigid and contravariantly finite. Let $\mathcal{W}$ be the class of morphisms $X \xrightarrow{f} Y$ such that, for any triangle $Z \xrightarrow{g} X \xrightarrow{f} Y \xrightarrow{h} \Sigma Z$, both morphisms g and h belong to the ideal $(T^{\perp})$.

Consider $J := \{0 \to \Sigma T\}$ and $I := \bigcup \mathscr{C}(R, A)$, where the union is taken over a set of representatives for the isomorphism classes of objects $R \in \operatorname{add} T$ and $A \in T * \Sigma T$. We define three classes of morphisms: $\mathcal{F}ib := J^{\square}$, $w\mathcal{F}ib := I^{\square}$ and $\mathcal{C}of := {}^{\square}w\mathcal{F}ib$.

Theorem 5.1.7. [Pal14, Theorem 2.2] *Let $\mathscr{C}$ be a triangulated category and let $T \in \mathscr{C}$ be rigid and contravariantly finite. Then the datum of $(\mathcal{W}, \mathcal{F}ib, \mathcal{C}of)$ is* almost *a model category structure on $\mathscr{C}$. Moreover:*

(i) All objects are fibrant.

(ii) An object is cofibrant if and only if it belongs to $T * \Sigma T$.

(iii) Two morphisms of fibrant and cofibrant objects are homotopic if and only if their difference factors through $\operatorname{add} \Sigma T$.

There are two reasons for the appearance of the term *almost* in the statement above. First, the category $\mathscr{C}$ does not have finite (let alone all small) limits and colomits in general: It only has finite direct sums, weak kernels and weak cokernels. However, these are enough in order to deduce the equivalence of categories in Theorem 5.1.6. Second, every morphism can be factored out as a trivial cofibration followed by a fibration, but the second factorisation only exists for morphisms with cofibrant domain. Again, this causes no trouble because any object is weakly equivalent to some cofibrant object.

As a consequence of the model structure, we obtain a homotopical interpretation of Buan–Marsh's result: it is an application of Quillen's Theorem 5.1.6!

Corollary 5.1.8. *The inclusion of $T * \Sigma T$ into $\mathscr{C}$ induces an equivalence of categories from $T * \Sigma T/(\Sigma T)$ to $\mathscr{C}[\mathcal{W}^{-1}]$. In particular, the localisation of $\mathscr{C}$ at $\mathcal{W}$ exists.*

We also give an homotopical interpretation of their key lemma:

Corollary 5.1.9. *The morphism $A \to X$ of Lemma 5.1.5 is a cofibrant replacement.*

Remark 5.1.10. In [Bel13, Lemma 4.4 and Theorem 4.6], Apostolos Beligiannis proved a generalised version of Theorem 5.1.4, which applies to contravariantly finite, rigid subcategories. This is also the generality in which we prove Theorem 5.1.7.

Remark 5.1.11. The localisation $\mathscr{C} \to \mathscr{C}[\mathcal{W}^{-1}]$ does not admit a calculus of fractions. However:

1. It is shown by Aslak Buan and Robert Marsh in [BM12] that the localisation functor factors as $\mathscr{C} \to \mathscr{C}/(\Sigma T) \to (\mathscr{C}/(\Sigma T))[\mathcal{W}^{-1}]$, where the second functor is a localisation that admits a calculus of fractions. This should be interpreted as $\mathscr{C} \to \mathscr{C}[\mathcal{W}^{-1}]$ having a calculus of fractions *up to homotopy*.

2. This calculus of fractions up to homotopy was proven, for exact categories, by Lucie Jacquet-Malo in her PhD thesis [JM17], by a purely homotopical algebraic approach, using a slight modification of the notion of a prefibration category in the sense of Radulescu-Banu [RB06].

5.2 Hovey's correspondence in extriangulated categories.

Hovey's correspondence [Hov02, Hov07] is a device for constructing model category structures on abelian categories. It is inspired by the somewhat canonical model category structure on a Frobenius category, but with two cotorsion pairs mimicking the role played by the projectives and the injectives. A similar, but slightly weaker correspondence, due to Apostolos Beligiannis and Idun Reiten and involving only one cotorsion pair appeared independently in [BR07, Theorem 4.2] (we note however, that their Theorem 3.5 suggests that some stronger statement should hold). Hovey's correspondence was generalised to exact categories by James Gillespie [Gil11, Gil15, Gil16]. He also discovered that this correspondence was a powerful tool for constructing model structures on categories of complexes, with the derived category as its homotopy category. The strategy of passing from a cotorsion pair on an exact category to a model structure on the category of complexes has recently been considered, and beautifully generalised, by Henrik Holm and Peter Jørgensen [HJ19]. In that article, the category of complexes is thought of as a category of representations of the A_∞^∞ quiver with radical square zero relations. This point of view allow them to generalise Gillespie's result to categories of representations of self-injective bound quivers, thus including categories of n-term complexes, of periodic complexes and many more. We also note the use of Gillespie's result in order to define the stable category of an arbitrary ring in [BGH14]

Hovey's correpondence has also been generalised in a different direction: the proof more or less applies as such to triangulated categories, with short exact sequences being replaced by triangles [Yan15].

5.2.1 Hovey's correspondence.

A model structure is equivalently described as two "intertwined" weak factorisation systems.

Definition 5.2.1. A *weak factorisation system* on a category $\mathscr{C}$ is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms in $\mathscr{C}$ satisfying:

1. The classes $\mathcal{L}$ and $\mathcal{R}$ are stable under retracts;

2. Any morphism $f \in \mathscr{C}$ factorises as $f = r \circ l$, for some $r \in \mathcal{R}$ and $l \in \mathcal{L}$;

3. The classes $\mathcal{L}$ and $\mathcal{R}$ are weakly orthogonal in the sense that any commutative square
$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "l"'] & X \arrow[d, "r"] \\ B \arrow[r, "\beta"'] \arrow[ur, dashed, "\gamma"] & Y, \end{tikzcd}$$
with $l \in \mathcal{L}$ and $r \in \mathcal{R}$, admits a lifting γ making both triangles commute.

Notation 5.2.2. We write $l \,\square\, r$ if, for any morphisms α, β such that the square above commutes, there is a lifting γ. We also write $\mathcal{L} \,\square\, \mathcal{R}$ when $l \,\square\, r$ for any $(l, r) \in \mathcal{L} \times \mathcal{R}$.

The strong link between weak factorisation systems and cotorsion pairs arises from the following:

Lemma 5.2.3. *Let $\mathscr{E}$ be an exact category and let $A \overset{f}{\rightarrowtail} B \twoheadrightarrow C$, $K \rightarrowtail X \overset{g}{\twoheadrightarrow} Y$ be two conflations (= short exact sequences). Assume that $\operatorname{Ext}^1_{\mathscr{E}}(C, K) = 0$. Then $f \,\square\, g$.*

Definition 5.2.4. A (complete) *cotorsion pair* in an exact category is a pair $(\mathcal{U}, \mathcal{V})$ of full subcategories such that, for any $X \in \mathscr{E}$, the following holds:

1. We have $X \in \mathcal{V}$ if and only if for any $U \in \mathcal{U}$, $\operatorname{Ext}^1_{\mathscr{E}}(U, X) = 0$.

2. We have $X \in \mathcal{U}$ if and only if for any $V \in \mathcal{V}$, $\operatorname{Ext}^1_{\mathscr{E}}(X, V) = 0$.

3. There are conflations $V_X \rightarrowtail U_X \twoheadrightarrow X$ and $X \rightarrowtail V^X \twoheadrightarrow U^X$ with $U_X, U^X \in \mathcal{U}$ and $V_X, V^X \in \mathcal{V}$.

The following is mostly a consequence of Lemma 5.2.3.

Proposition 5.2.5. *Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair on an exact category $\mathscr{E}$. Define $\mathcal{L}$ to be the class of all those inflations whose cokernel belongs to $\mathcal{U}$. Dually, define $\mathcal{R}$ to be the class of all those deflations whose kernel belongs to $\mathcal{V}$. Then $(\mathcal{L}, \mathcal{R})$ is a weak factorisation system.*

Remark 5.2.6. In a work (not) in progress with Peter Jørgensen, we proved a version adapted to higher homological algebra: Let $\mathscr{C}$ be an n-angulated category with suspension functor Σ (a similar statement holds in the n-exact case), and let $A \xrightarrow{f} B \to C^* \to \Sigma A$ and $X \xrightarrow{g} Y \to Z^* \to \Sigma X$ be two n-angles. Then $f \,\square\, g$ if and only if any morphism of complexes $C^* \to Z^*$ is null-homotopic.

Let $\mathscr{C}$ be a category with finite products and coproducts. Then a model structure on $\mathscr{C}$ is equivalently defined as three classes of morphisms $(\mathcal{F}ib, \mathcal{C}of, \mathbb{W})$ such that $(\mathcal{C}of \cap \mathbb{W}, \mathcal{F}ib)$ and $(\mathcal{C}of, \mathcal{F}ib \cap \mathbb{W})$ are weak factorisation systems and the class $\mathbb{W}$ satisfies the two-out-of-three condition. This was first noticed by André Joyal and Myles Tierney when $\mathscr{C}$ either has push-outs or has pull-backs (see also [MP12, Lemma 14.2.5]). The general case follows from [Egg06] and the fact that weak equivalences are precisely those morphisms that become isomorphisms in the localisation $\mathscr{C}[\mathbb{W}^{-1}]$. This is explained in detail in Pierre Cagne's PhD Thesis [Cag18, Section 2.2].

Question 5.2.7. Let $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$ be two cotorsion pairs on an exact category $\mathscr{E}$. We obtain, by Proposition 5.2.5, two weak factorisation systems $(w\mathcal{C}of, \mathcal{F}ib)$, $(\mathcal{C}of, w\mathcal{F}ib)$. Define $\mathbb{W} = w\mathcal{F}ib \circ w\mathcal{C}of$. Under which conditions on the two cotorsion pairs is $(\mathcal{F}ib, \mathcal{C}of, \mathbb{W})$ a model structure on $\mathscr{E}$?

Note that if $(\mathcal{F}ib, \mathcal{C}of, \mathbb{W})$ is a model structure on $\mathscr{E}$, then an object is cofibrant (resp. fibrant, trivially cofibrant, trivially fibrant) if and only if it belongs to $\mathcal{U}$ (resp. to $\mathcal{T}$, $\mathcal{S}$, $\mathcal{V}$). Part of the answer to Question 5.2.7 is then rather immediate: First, it is necessary that $\mathcal{S} \subseteq \mathcal{U}$ and $\mathcal{V} \subseteq \mathcal{T}$ (those two conditions are equivalent). Second, both $\mathcal{U} \cap \mathcal{V}$ and $\mathcal{S} \cap \mathcal{T}$ should be the full subcategory of trivially cofibrant and trivially fibrant objects, hence it is necessary that $\mathcal{U} \cap \mathcal{V} = \mathcal{S} \cap \mathcal{T}$. The last condition is less obvious, and arises when considering the following question: Is it possible to read the two-out-of-three property for $\mathbb{W}$ directly on the two cotorsion pairs? Some very specific instance of the two-out-of-three property implies that, for any object X in $\mathscr{E}$, the morphism $0 \to X$ is a weak equivalence if and only if so is $X \to 0$. This easily translate to a condition on $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$. It turns out that this condition implies the two-out-of-three property for $\mathbb{W}$ in full generality.

Definition 5.2.8. Two cotorsion pairs $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$ on an exact category $\mathscr{E}$ are called:

1. *Twin* cotorsion pairs if $\mathcal{S} \subseteq \mathcal{U}$ (or equivalently $\mathcal{V} \subseteq \mathcal{T}$);
2. *Concentric* twin cotorsion pairs if moreover $\mathcal{U} \cap \mathcal{V} = \mathcal{S} \cap \mathcal{T}$;
3. *Hovey twin cotorsion pairs* if moreover, for any object X, the existence of a conflation $V \rightarrowtail S \twoheadrightarrow X$ with $V \in \mathcal{V}$, $S \in \mathcal{S}$ is equivalent to the existence of a conflation $X \rightarrowtail V' \twoheadrightarrow S'$ with $V' \in \mathcal{V}$, $S' \in \mathcal{S}$.

Remark 5.2.9. Any Hovey twin cotorsion pairs are concentric.

Not all model structures on $\mathscr{E}$ can be expected to come from twin cotorsion pairs (for counter-examples, see the model structures arising from rigid objects in [JM17]). However, all those that interact nicely with the exact structure do.

Definition 5.2.10. A model structure on $\mathscr{E}$ is called an *exact model structure* if cofibrations are precisely those inflations whose cokernel is cofibrant and fibrations are precisely those deflations whose kernel is fibrant.

In an exact model structure, the acyclic cofibrations coincide with the inflations having a trivially cofibrant cokernel, and the acyclic fibrations coincide with the deflations having a trivially fibrant kernel.

For some technical reasons, Gillespie's version of Hovey's correspondence requires the exact category to be weakly idempotent complete:

Definition 5.2.11. An additive category is *weakly idempotent complete* if any section has a cokernel and any retraction has a kernel. For exact categories, those two conditions are equivalent and also equivalent to the fact that if $g \circ f$ is a deflation (resp. an inflation) then g is a deflation (resp. f is an inflation).

We are now ready to give a slightly reformulated statement of Gillespie's generalisation to exact categories of Hovey's correpondence.

Theorem 5.2.12 (Hovey; Gillespie)**.** *Let $\mathscr{E}$ be a weakly idempotent complete exact category. Then there is a bijective correspondence between exact model structures on $\mathscr{E}$ and Hovey twin cotorsion pairs on $\mathscr{E}$.*

If $(\mathcal{F}ib, \mathcal{C}of, \mathbb{W})$ is an exact model structure, the associated cotorsion pairs are given by (*wcof,fib*), (*cof,wfib*) where *cof* (resp *fib, wcof, wfib*) is the class of cofibrant (resp. fibrant, trivially cofibrant, trivially fibrant) objects. Conversely, if $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$ are Hovey twin cotorsion pairs, the associated $(\mathcal{F}ib, \mathcal{C}of, \mathbb{W})$ is given by the following recipe: a morphism is a fibration (resp. cofibration) if and only if it is a deflation with kernel in $\mathcal{T}$ (resp. an inflation with cokernel in $\mathcal{U}$) and a morphism is in $\mathbb{W}$ if and only if it factorises as an inflation with cokernel in $\mathcal{S}$ followed by a deflation with kernel in $\mathcal{V}$.

5.2.2 Homotopy categories of exact model categories.

In [NP19], it is shown that Hovey's correspondence also holds for extriangulated categories. Almost all the definitions of the previous section have obvious analogues for extriangulated categories. The only exception is weak idempotent completeness, which does not seem to be equivalent to the characterisation which is used in Gillespie's proof of Hovey's correspondence for exact categories.

We refer to Section 4.1 for an introduction to the language of extriangulated categories.

Definition 5.2.13. An extriangulated category $\mathscr{C}$ is said to satisfy condition (WIC) if, for any pair of composable morphisms (f, g), if $g \circ f$ is an inflation, then f is an inflation and if $g \circ f$ is a deflation, then g is a deflation.

In that setting, we call *admissible* model structure, the analogue of what was called an exact model structure in the previous section.

Theorem 5.2.14. [NP19, Proposition 5.6 and Section 5.3] *Let $\mathscr{C}$ be an extriangulated category satisfying condition* (WIC)*. Then there is a bijective correspondence between admissible model structures on $\mathscr{C}$ and Hovey twin cotorsion pairs on $\mathscr{C}$.*

The recipe for constructing a model structure out of some Hovey twin cotorsion pairs is similar to that for exact categories.

Corollary 5.2.15. *Let $\mathscr{C}$ be an extriangulated category satisfying condition* (WIC)*, and let $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$ be Hovey twin cotorsion pairs on $\mathscr{C}$. Then the inclusion $\mathcal{T} \cap \mathcal{U} \to \mathscr{C}$ induces an equivalence of categories*

$$\mathcal{T} \cap \mathcal{U} \,/\, (\mathcal{S} \cap \mathcal{V}) \xrightarrow{\simeq} \mathscr{C}[\mathbb{W}^{-1}]$$

expressing the localisation of $\mathscr{C}$ at the class $\mathbb{W}$ as an ideal subquotient of $\mathscr{C}$.

Remark 5.2.16. In the specific case when $\mathscr{C}$ is triangulated, $(\mathcal{S},\mathcal{T})=(\mathcal{U},\mathcal{V})$ and the full subcategory of $\mathscr{C}$ whose objects are weakly equivalent to 0 is thick (this latter condition is related to the notion of a hereditary cotorsion pair, defined below), Corollary 5.2.15 recovers an independant result of Osamu Iyama and Dong Yang [IY17, Theorem 1.1], which is inspired by, and generalises Amiot–Guo–Keller's equivalence appearing in their constructions of (higher, generalised) cluster categories.

Corollary 5.2.15 gives a strategy for studying the structure of the homotopy category $\mathscr{C}[\mathbb{W}^{-1}]$.

Definition 5.2.17. A cotorsion pair $(\mathcal{X},\mathcal{Y})$ of an exact category $\mathscr{E}$ is called *hereditary* if $\mathcal{X}$ is stable under taking kernels of epimorphisms (between objects in $\mathcal{X}$) and $\mathcal{Y}$ is stable under taking cokernels of monomorphisms (between objects in $\mathcal{Y}$).

Proposition 5.2.18 (Gillespie)**.** *Let $\mathscr{E}$ be an exact category with some Hovey twin cotorsion pairs $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$. Assume that $(\mathcal{S},\mathcal{T})$ and $(\mathcal{U},\mathcal{V})$ are hereditary. Then the homotopy category $\mathscr{C}[\mathbb{W}^{-1}]$, for the exact model structure obtained by Hovey's correspondence, is triangulated.*

Proof. When $(\mathcal{S},\mathcal{T})$ and $(\mathcal{U},\mathcal{V})$ are hereditary, the category $\mathcal{T}\cap\mathcal{U}$ is stable under extensions in $\mathscr{E}$, hence exact. Moreover, it is easily seen to be Frobenius, with projective-injective objects $\mathcal{S}\cap\mathcal{V}$. Hence $\mathcal{T}\cap\mathcal{U}/(\mathcal{S}\cap\mathcal{V})$ is triangulated, and so is $\mathscr{C}[\mathbb{W}^{-1}]$ by Corollary 5.2.15. □

We give a generalisation of this result, which seems to be new already in the case of exact categories.

Theorem 5.2.19. [NP19, Theorem 6.20] *For any extriangulated category $\mathscr{C}$ satisfying condition* (WIC) *and any Hovey twin cotorsion pairs $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$ on $\mathscr{C}$, the associated homotopy category $\mathscr{C}[\mathbb{W}^{-1}]$ is triangulated.*

Remark 5.2.20. Under the assumptions of Theorem 5.2.19, the model structure on $\mathscr{C}$ is not stable *stricto sensu*. However, the equivalent of the octahedral axiom gives specific choices of weak bicartesian squares that compensate for the lack of stability. It might be interesting to investigate in which sense those model structures might be considered stable.

A specific case of Theorem 5.2.19, Corollary 5.2.22 below, is worth mentionning as it shows that Theorem 5.2.19 generalises two well-known results:

- When applied to Frobenius exact categories, it recovers Happel's theorem that the stable category is triangulated [Hap88].
- When suitably applied to triangulated categories, it recovers Iyama–Yoshino reduction [IY08].

Definition 5.2.21. A *Frobenius extriangulated category* is an extriangulated category $(\mathscr{C},\mathbb{E},\mathfrak{s})$ which has enough projectives and enough injectives and such that $\mathrm{Proj}(\mathscr{C})=\mathrm{Inj}(\mathscr{C})$.

Corollary 5.2.22. *Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be a Frobenius extriangulated category satisfying condition (WIC). Then its stable category $\mathscr{C}/\big(\mathrm{Proj}(\mathscr{C})\big)$ is triangulated.*

Remark 5.2.23. Because this result is given as a corollary of Theorem 5.2.19, our proof makes use of the model structure, and in particular of the fact that the homotopy category is given as a localisation of $\mathscr{C}$. However, it is possible to give a direct proof, mimicking that of Happel (or that of Iyama–Yoshino), which shows that assumption (WIC) is not needed. This approach has been carefully carried out by Owen Garnier in his Master 1 thesis.

We conclude this section with the hope that extriangulated categories might be related to higher category theory.

Remark 5.2.24. There is a notion of exact infinity-categories [Bar15, Bar13], whose homotopy categories are expected to be extriangulated (work in progress with Hiroyuki Nakaoka). One might thus wonder if there exists a version of Hovey's correpondence for exact infinity-categories, enhancing Theorem 5.2.14.

5.2.3 Mutations of twin cotorsion pairs.

That cotorsion pairs form a good setup for studying mutations, even of cluster tilting subcategories, goes back to [IY08]. In this section, we show that once we have fixed some Hovey twin cotorsion pairs on $\mathscr{C}$, there is a notion of mutation for cotorsion pairs on $\mathscr{C}$. We note that not all cotorsion pairs can be mutated (whence the notion of a *mutable* cotorsion pair) and that mutation is not an involution in general.

Let $\mathscr{C}$ be an extriangulated category satisfying condition (WIC), and let $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be Hovey twin cotorsion pairs on $\mathscr{C}$. Write $\widetilde{\mathscr{C}}$ for the localisation of $\mathscr{C}$ with respect to the weak equivalences for the model structure associated with $\mathcal{P}$, and $\ell : \mathscr{C} \to \widetilde{\mathscr{C}}$ for the localisation functor. By Theorem 5.2.19, the category $\widetilde{\mathscr{C}}$ is triangulated.

Definition 5.2.25. The class of *mutable cotorsion pairs* on $\mathscr{C}$ endowed with $\mathcal{P}$ is

$$\mathfrak{M}_{\mathcal{P}} = \left\{ (\mathcal{A}, \mathcal{B}) \in \mathfrak{CP}(\mathscr{C}) \;\middle|\; \begin{matrix} \mathcal{S} \subseteq \mathcal{A} \subseteq \mathcal{U} \\ \mathcal{V} \subseteq \mathcal{B} \subseteq \mathcal{T} \end{matrix} \;,\; \mathrm{Ext}^1_{\widetilde{\mathscr{C}}}(\ell(\mathcal{A}), \ell(\mathcal{B})) = 0 \right\}.$$

Remark that the conditions in the definition of a mutable cotorsion pair are redundent. Indeed, the conditions $\mathcal{S} \subseteq \mathcal{A}$ and $\mathcal{B} \subseteq \mathcal{T}$ are equivalent, and similarly for $\mathcal{A} \subseteq \mathcal{U}$ and $\mathcal{V} \subseteq \mathcal{B}$.

The main step towards defining some mutation is to show that mutable cotorsion pairs in $\mathscr{C}$ correspond precisely to cotorsion pairs in the localisation $\widetilde{\mathscr{C}}$.

Theorem 5.2.26. *Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category satisfying condition (WIC), and let $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be Hovey twin cotorsion pairs on $\mathscr{C}$. Sending a mutable cotorsion pair $(\mathcal{A}, \mathcal{B})$ to the pair $\mathbb{R}((\mathcal{A}, \mathcal{B})) = (\ell((A), \ell(\mathcal{B}))$ and the cotorsion pair $(\mathcal{L}, \mathcal{R})$ to the pair $\mathbb{I}((\mathcal{L}, \mathcal{R})) = (\mathcal{U} \cap \ell^{-1}(\mathcal{L}), \mathcal{T} \cap \ell^{-1}(\mathcal{R}))$ gives mutually inverse bijective correspondences between mutable cotorsion pairs on $\mathscr{C}$ and cotorsion pairs on $\widetilde{(\mathscr{C})}$.*

$$\mathbb{R} : \mathfrak{M}_{\mathcal{P}} \overset{1:1}{\longleftrightarrow} \mathfrak{CP}(\widetilde{\mathscr{C}}) : \mathbb{I}$$

Theorem 5.2.26 allows us to define a notion of mutation for mutable cotorsion pairs, induced by the shift functor Σ on the triangulated category $\widetilde{\mathscr{C}}$.

Definition 5.2.27. Under the assumptions of Theorem 5.2.26, let $(\mathcal{A}, \mathcal{B}) \in \mathfrak{M}_{\mathcal{P}}$ be a mutable cotorsion pair. The *mutation* of $(\mathcal{A}, \mathcal{B})$ *with respect to* $(\mathcal{S}, \mathcal{V})$ is the mutable cotorsion pair

$$\mu(\mathcal{A}, \mathcal{B}) = \mathbb{I} \circ \Sigma \circ \mathbb{R}\,(\mathcal{A}, \mathcal{B}) = \big(\mathcal{U} \cap \ell^{-1}(\Sigma\,\ell(\mathcal{A})), \mathcal{T} \cap \ell^{-1}(\Sigma\,\ell(\mathcal{B}))\big).$$

This definition of mutation is inspired from Osamu Iyama and Yuji Yoshino's approach to the mutation of cluster tilting objects in [IY08]. It aims at generalising various notions of mutation that appear in the literature: mutation of cluster tilting objects, of 2-term silting objects, of (intermediate bounded) t-structures and of (intermediate bounded) co-t-structures. For that purpose, it would be interesting to generalise mutation with respect to $\mathcal{P}$ from the case of Hovey twin cotorsion pairs to that of concentric twin cotorsion pairs.

The results in Section 5.2 rely heavily on additivity. However, many model categories arising in nature are not additive. This raises the question of defining a non-additive version of cotorsion pairs:

Question 5.2.28. Is there a version of Hovey's correspondence adapted for the (non-additive) proto-exact categories of [DK12] ?

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